University of California, Los Angeles
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Instructor: T. Arant
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Name: [redacted] UCLA ID: [redacted] Signature: [redacted]

MATH 131A: Real Analysis
Midterm 2

This exam contains 6 pages (including this cover page) and 4 problems. Your solutions to the problems must be uploaded to Gradescope before 8am PST on February 26, 2022.

This is a take-home exam. The following rules regarding the take-home format apply:

- The exam is an open-book/open-notes/open-internet exam.

  You **cannot** collaborate in any way with any individual on the exam. Any form of communication/consultation/collaboration with another person about the exam is expressly prohibited—this includes, but is not limited to, Zoom meetings, email, telephone calls, texting, making posts on stack exchanges, etc. Violation of the no-collaboration policy is a violation of the UCLA code of student conduct and will come with serious consequences.
- The instructor reserves the right to ask any student for clarification regarding any of the student's exam answers at any time during a two week period after the day of the exam. This may require a Zoom meeting with the instructor.
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You are required to show your work on each problem of this exam. The following rules apply:

- **All answers must be justified. Mysterious or unsupported answers will not receive credit**. A correct answer, unsupported by carefully written proof will receive no credit; an incorrect answer supported by substantially correct explanations might still receive partial credit.
- **If you use a theorem or proposition from class or the notes or the textbook or a result established in the homework, you must indicate this and explain why the** theorem may be applied.
- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

Good luck!

Academic integrity pledge:

Upon my honor, I affirm that I did not solicit nor did I receive the help of any individual in writing my answers to this exam.

Signature: [redacted]

Print name: [redacted]

1. (10 points) Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that if (s_n) is a Cauchy sequence in $[a, b]$, then $(f(s_n))$ is convergent.

Proof. First note that $a<b \Rightarrow [a,b]$ is nonempty.
When we discussed Cauchy and monotone sequences it was discussed in both ~~class~~ the textbook and lecture that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. (Thm 10.11 from textbook).

Let (s_n) be a Cauchy sequence in $[a,b]$. (i.e. $\forall n \in \mathbb{N}$, $s_n \in [a,b]$).
$\therefore (s_n)$ is convergent, ~~i.e. $\lim s_n = s$ where $s \in \mathbb{R}$~~
i.e. $\exists\, s \in \mathbb{R}$ s.t. $\lim s_n = s$.

Now we show that $s \in [a,b]$. Suppose for contradiction that $s > b$. We know that since ~~$\lim s_n = s$~~ $\lim s_n = s$, we have that

$\forall \varepsilon > 0$, $\exists\, N$ s.t. $n > N \Rightarrow |s_n - s| < \varepsilon$.

Let $\varepsilon = s - b$. (since $s > b$, we know $\varepsilon > 0$).

$\therefore \exists\, N_1$ s.t. $n > N_1 \Rightarrow |s_n - s| < s - b$

$\Rightarrow$ ~~$s_n - s$~~ $s_n - s > -(s-b)$

$\Rightarrow s_n > s - s + b \Rightarrow s_n > b$

i.e. for $n > N_1$, s_n would not be in the interval $[a,b]$.

Now suppose for contradiction that $s < a$. Let $\varepsilon =$ ~~$s-a$~~ $a - s$ ($\varepsilon > 0$, since $a > s$).

$\therefore \exists\, N_2$ s.t. $n > N_2 \Rightarrow |s_n - s| < a - s$

$\Rightarrow s_n - s < a - s \Rightarrow s_n < a$

i.e. that for $n > N_2$, s_n would not be in the interval $[a,b]$.

Thus, $s > b$ and $s < a$ are both false, $\therefore$ $s \in [a,b]$.

From the question statement we know that f is continuous, i.e. continuous over its domain $[a,b]$. $\therefore$ for every $x \in [a,b]$, f is defined and continuous at x.
$\therefore$ Since $s \in [a,b]$, f is continuous at s. $\therefore$ By the definition of continuity at a point, since $\lim s_n = s$, it follows that

$\lim (f(s_n)) = f(s)$. $\therefore$ The sequence $(f(s_n))$ is convergent, namely to $f(s)$. $\blacksquare$

2. (12 points) Let $M \in \mathbb{R}$ and let (s_n) be a sequence satisfying both of the following conditions:

   (i) $s_n < M$ for all $n \in \mathbb{N}$;

   (ii) for every $\epsilon > 0$, the set $\{n \in \mathbb{N} : |s_n - M| < \epsilon\}$ is infinite.

   Prove that $\limsup s_n = M$.

Proof.

Consider the fact that, given the sequence (s_n), we have that

$$\forall \varepsilon > 0, \quad \{n \in \mathbb{N} : |s_n - M| < \varepsilon\} \text{ is infinite.}$$

It was discussed in both lecture and the textbook that a ~~non~~ real number t can be a subsequential limit of any (s_n) in $\mathbb{R}$ (i.e. there can exist some subsequence (s_{n_k}) converging to t) ~~exists~~ if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite for all ~~$\varepsilon > 0$~~ $\varepsilon > 0$. (Thm 11.2 of textbook)

$\therefore$ Given property (ii) of (s_n) that $\forall \varepsilon > 0$, $\{n \in \mathbb{N} : |s_n - M| < \varepsilon\}$ is infinite ~~and~~ (and that $M \in \mathbb{R}$), it follows that M is a subsequential limit of ~~(s_n)~~ (s_n).

$\therefore$ ~~$\exists$~~ There is a subsequence of ~~$(s$~~ (s_n) converging to M.

Now, consider the set S of all subsequential limits of (s_n).
We know that $M \in S$, i.e., M is a subsequential limit of (s_n).
Now consider any $s \in S$, i.e., any subsequential limit of (s_n)
We know there is some subsequence ~~(s_{n_k})~~ (s_{n_k}) of (s_n) ~~th~~ that converges to s, i.e. $\lim_{k \to \infty} s_{n_k} = s \in S$.

We further know that n_k is just some natural number m, for all k, i.e. ~~$n_k = m \in \mathbb{N} \ \forall k$~~ $n_k = m \in \mathbb{N} \ \forall k \in \mathbb{N}$, i.e., $\forall k \in \mathbb{N}$, s_{n_k} is some term s_m of (s_n).
We have from property (i) that $s_m < M \ \forall m \in \mathbb{N}$. $\therefore$ $s_{n_k} < M \ \forall k \in \mathbb{N}$.

~~Lemma: We show $s_m < M \ \forall m \in \mathbb{N}$, $s_m < M \ \forall m \in \mathbb{N} \Rightarrow \lim s_m$~~
~~Let (s_n) be a convergent sequence~~

Lemma: Let (t_n) be a convergent sequence in $\mathbb{R}$ and let $A \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $t_n < A$. ~~If~~ Then, $\lim t_n \leq A$.

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Problem 2 continued

Proof of Lemma: Suppose (t_n) converges to ~~$t \in \mathbb{R}$~~ $t \in \mathbb{R}$.
$\lim t_n = t$, and $t_n < A$ $\forall n \in \mathbb{N}$.
Suppose for contradiction that $\lim t_n > A$, i.e. $t > A$.
~~Then~~ Choose $\varepsilon = t - A$.
Since $\lim t_n = t$, then by the definition of the limit of a sequence, it follows that

$$\exists N \text{ s.t. } n > N \Rightarrow |t_n - t| < \varepsilon = t - A$$
$$\Rightarrow -\varepsilon < t_n - t < \varepsilon$$
$$\Rightarrow t - \varepsilon < t_n < t + \varepsilon$$
$$\Rightarrow t_n > t - \varepsilon \quad \Rightarrow t_n > t - (t - A) = A$$
$$\Rightarrow t_n > A.$$

$\therefore$ for all $n > N$, we have that ~~$t_n > A$ and~~ $t_n < A$ by the hypothesis and $t_n > A$, which is a contradiction.
$\therefore$ it cannot be true that $t = \lim t_n > A$
$\therefore$ $\lim t_n \leq A$. $\blacksquare$ (Lemma).

Now, back to ~~Problem 2~~ Problem 2, for any $s \in S$ that is a subsequential limit of (s_n), we know that there is some subsequence of (s_n), (s_{n_k}) that converges to s, i.e.

$$\lim_{k \to \infty} s_{n_k} = s$$

But, we also have that

$$\forall k \in \mathbb{N}, \quad s_{n_k} < M.$$

Therefore by the lemma, $\lim s_{n_k} \leq M$, i.e., $s \leq M$.
$\therefore$ For any subsequential limit s of (s_n), $s \leq M$. But M is also a subsequential limit of (s_n). Therefore, $\max S = M \Rightarrow \sup S = M$ by a theorem about the equality of maximum and supremum when the maximum exists.
$\therefore$ $\sup S = M$. Now, by Theorem 11.8 of the textbook, we have that $\lim \sup s_n = \sup S = M$. $\boxtimes$

3. Recall that for a series $\sum a_n$, the kth partial sum is

$$s_k = a_1 + a_2 + \cdots + a_k.$$

(a) (6 points) Give an example of a series $\sum a_n$ for which the sequence of partial sums (s_k) is bounded and $\sum a_n$ diverges.

Consider the sequence $(a_n)_{n \in \mathbb{N}}$ defined by $a_n = (-1)^n$

The ~~seies~~ series $\sum a_n$ has partials sums $(s_k)_{k \in \mathbb{N}}$ as follows:

$s_1 = (-1)$, $s_2 = (-1) + 1 = 0$

$s_3 = (-1) + 1 + (-1) = -1$, $s_4 = (-1) + 1 + (-1) + 1 = 0$.

It can be shown by induction that $\forall n \in \mathbb{N}$, $s_{2n} = 0$ and $s_{2n+1} = -1$

(Base case for s_{2n+1} when $n=0$: $s_1 = -1$. Consider $s_{2n+1} = -1$. Then $s_{2n+3} = s_{2n+1} + 1 - 1 = s_{2n+1} = -1$. Base case for s_{2n} when $n=1$: $s_2 = 0$. Consider $s_{2n+2} = s_{2n} + (-1) - 1 = s_{2n} = 0$. [Proof sketch])

$\therefore \forall k \in \mathbb{N}$, $-1 \leq s_k \leq 0$. Therefore the sequence of partial sums (s_k) is bounded.

However, consider Theorem 14.4 from the textbook: A series $\sum a_n$ is convergent $\Leftrightarrow$ it satisfies the Cauchy criterion

Cauchy criterion: $\forall \varepsilon > 0 \ \exists N$ s.t. $n \geq m > N \Rightarrow \left|\sum_{k=m}^{n} a_k\right| < \varepsilon$. Let $\varepsilon = 0.5$ and $n = m$

$\left|\sum_{k=m}^{m} (-1)^k\right| = |(-1)|$ or $|1| = 1$ which is not less than 0.5, and this holds $\forall m \in \mathbb{N}$. Thus, $\sum a_n$ ~~does~~ ~~not~~ is ~~bounded~~ (s_k) is bounded but $\sum a_n$ does not satisfy the Cauchy criterion, and so, diverges. ∎

(b) (8 points) Suppose that $\sum a_n$ is a series such that $a_n \geq 0$ for all $n \in \mathbb{N}$, and the sequence of partial sums (s_k) is bounded. Prove that $\sum a_n$ converges.

$a_n \geq 0 \ \forall n \in \mathbb{N}$ and (s_k) defined by $s_k = \sum_{n=1}^{k} a_n$ is bounded

$\sum a_n$ converges.

Proof.

(s_k) is bounded $\Rightarrow \exists M \in \mathbb{R}$ s.t. $|s_k| \leq M \ \forall k \in \mathbb{N}$

$\therefore -M \leq s_k \leq M \ \forall k \in \mathbb{N}$.

~~$\therefore \lim_{k\to\infty} s_k \geq \sum a_n \in [-M, M]$.~~

~~This so~~ Now, note that $a_n \geq 0 \ \forall n \in \mathbb{N}$.

$\therefore$ for any $k \in \mathbb{N}$, $s_{k+1} = s_k + a_{k+1} \geq s_k$. ~~$\therefore (s_k)$ is a monoton~~

$\therefore (s_k)$ is a ~~monotically~~ monotonically increasing sequence, and it is bounded.

$\therefore$ By the theorem that all bounded monotone sequences converge (Theorem 10.2 in textbook)

it follows that (s_k) converges to some real number $s \in \mathbb{R}$.

$\therefore \lim_{k\to\infty} s_k \in \mathbb{R}$. But $\lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n =$ ~~$\sum a_n \in \mathbb{R}$~~ is defined to be $\sum a_n$. $\therefore \sum a_n$ converges. ∎

4. (a) (6 points) Show that for any $a, b \in \mathbb{R}$,

$$\min\{a, b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|.$$

Prof.

Case 1: ~~a~~ $a \geq b$. In this case, ~~$a \geq b$~~ $a - b \geq 0$ $\therefore |a-b| = a-b$.

$\therefore \frac{1}{2}(a+b) - \frac{1}{2}|a-b| = \cancel{\frac{1}{2}a + \frac{1}{2}}\ \frac{1}{2}(a+b) - \frac{1}{2}(a-b)$

$= \frac{1}{2}(a+b-a+b) = \frac{1}{2}(2b) = b.$

Clearly, since ~~$a \geq b$~~ $a \geq b$, $\min\{a,b\} = b$ (in the case $a=b$, we can choose either a or b as the min so it doesn't matter that b is chosen as the min in this case)

$\therefore$ ~~ta~~ in this case, ~~$\min\{a,b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a$~~

$\min\{a,b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|$ and we are done.

Case 2: $a < b$. In this case $a - b < 0$. $\therefore |a-b| = -(a-b) = b-a$

$\therefore \frac{1}{2}(a+b) - \frac{1}{2}|a-b| = \frac{1}{2}(a+b) - \frac{1}{2}(b-a) = \frac{1}{2}(a+b-b+a) = \frac{1}{2}(2a) = a$

Clearly since $a<b$, $\min\{a,b\} = a$. $\therefore$ In this case as well, $\min\{a,b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|$. ∎

(b) (8 points) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions, then the function

$$h : \mathbb{R} \to \mathbb{R}, \qquad h(x) = \min\{f(x), g(x)\}$$

is also continuous. *Hint.* Use part (a).

Prof.

It was proven in ~~class~~ lecture and the textbook that for any continuous function ~~f with~~ $f : \mathrm{dom}(f) \to \mathbb{R}$ where $\mathrm{dom}(f)$ ~~≠~~ $\subseteq \mathbb{R}$, the functions $|f|$ and (kf) where $k \in \mathbb{R}$ are continuous. (Thm 17.3 of textbook)

Furthermore for any continuous functions $f : \mathrm{dom}(f) \to \mathbb{R}$, $g : \mathrm{dom}(g) \to \mathbb{R}$, the function $(f+g) : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R}$ is also continuous.

From part ⓐ: $h(x) = \min\{f(x), g(x)\} = \frac{1}{2}(f(x)+g(x)) - \frac{1}{2}|f(x)-g(x)|$

$f(x)+g(x) = (f+g)(x)$, ~~which~~ and $(f+g)$ is continuous since f and g are.

$(-g$ ~~(x)~~$)$ is continuous since g is. $\therefore f(x) - g(x) = (f+(-g))(x)$, and $(f+(-g))$ is continuous since f and $(-g)$ are. $|(f-g)(x)|$ is continuous at all $x \in \mathbb{R}$, since $(f-g)$ is continuous, and absolute values of continuous functions are continuous.

$\frac{1}{2}(f+g)$ and $\frac{1}{2}|f-g|$ are continuous since constant multiples of continuous functions are continuous.

$\therefore \frac{1}{2}(f+g) + \frac{1}{2}|f-g| = \min\{f,g\}$ is continuous because $\frac{1}{2}(f+g)$ and $\frac{1}{2}|f-g|$ are, and sums of functions are continuous. Note that all functions are defined from $\mathbb{R} \to \mathbb{R}$ since ~~Bec~~ $\mathrm{dom}(f) = \mathrm{dom}(g) = \mathbb{R}$ and $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$. $\therefore$ ~~h~~ $h : \mathbb{R} \to \mathbb{R}$ is continuous. ∎