Discussion and Solutions for Midterm II

There were three recurring errors on this exam:

(A). "$\lim a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges" This is simply not true – *as question 1(i) on this exam shows.* You are confusing this statement with its converse "$\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim a_n = 0$" – which is true and useful.

(B) Misusing the comparison test when series do not have positive terms. The hypothesis in the comparison test is $|a_n| \leq b_n$ when $n > N$, **not** $a_n \leq b_n$ when $n > N$. If the a_n's are not all positive, $a_n \leq b_n$ tells you nothing about how big $|a_n|$ is.

(C) People had trouble writing partial sums correctly: You have to write $s_n = \sum_{k=1}^{n} a_k$. Otherwise you end up with $s_n = \sum_{n=1}^{n} a_n$, where there are too many n's, or $s_n = \sum_{n=1}^{\infty} a_n$ which is not a partial sum.

In the solutions here I will give the shortest answers. Other answers can be correct, but they usually are not.

1. Determine which of the following series converge and state which test or theorem you would use to prove your answer. You do not need to carry out the tests, but for comparison tests state which series you would use for comparison.

(i) (3 pts.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Answer: This series diverges by comparison with $\sum \frac{1}{n}$, since that series diverges and $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$.

(ii) (3pts.) $\sum_{n=1}^{\infty} \frac{n^2}{2^n - n}$

Answer: This series converges by the ratio test. While you were not asked to carry out the test, many people tried and failed. It goes this way:

$$|\frac{a_{n+1}}{a_n}| = \frac{(n+1)^2}{2^{n+1} - (n+1)} \frac{2^n - n}{n^2} = \frac{1}{2}(1 + \frac{1}{n})^2 (\frac{1 - n2^{-n}}{1 - (n+1)2^{-n}})$$

which has limit 1/2 as $n \to \infty$ – remember that $\lim_{n\to\infty} \frac{n^p}{a^n} = 0$ for all powers p when $a > 1$.

(iii) (4 pts.) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$

Answer: This converges by the Alternating Series Theorem – it's a theorem instead of a test.

2. The instructions for this one are the same as the instructions for problem 1 – except that this time I am asking you to carry out the tests.

(iv) (3 pts.) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4-2n^2+6}$

Answer: This one converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, but people had trouble doing the comparison. The limit comparison test works best: Let $a_n = \frac{n^2+n+1}{n^4-2n^2+6}$ and $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4+n^3+n^2}{n^4-2n^2+6} = \lim_{n\to\infty} \frac{1+n^{-1}+n^{-2}}{1-2n^{-2}+6n^{-4}} = 1.$$

(v) (3 pts.) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n-1}$

Answer: This diverges because $\lim_{n\to\infty} |a_n| = 1/2$.

(vi) (4 pts.) $\sum_{n=1}^{\infty} \frac{n^n}{4^n n!}$

Answer: For this one you had to use the ratio test. If you tried the root test, you ended up with $\lim \frac{n}{4(n!)^{1/n}}$ which cannot be computed with anything from this or most other undergraduate courses. Anyway

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{(n+1)^n}{4n^n} = \frac{1}{4} \lim_{n\to\infty} (1+\frac{1}{n})^n = \frac{e}{4} < 1$$

So this converges. $\lim_{n\to\infty}(1+\frac{1}{n})^n$ turned up in HW Problem 27, but very few people remembered it, and there were many interesting conjectures about what $\lim_{n\to\infty}(1+\frac{1}{n})^n$ might be.

3. (10 pts.) The sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and increasing: $a_n \le a_{n+1}$. Prove that there is an $\alpha \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = \alpha$.

Proof: This problem appeared – for a *decreasing* sequence – as Problem 3 on the *first* hour exam. It could appear again, be warned.

Since $S = \{a_n : n \in \mathbb{N}\}$ is a set of real numbers that is bounded above, it has a least upper bound $\alpha \in \mathbb{R}$. So $a_n \le \alpha$ for all n. Given $\epsilon > 0$, $\alpha - \epsilon$ is not an upper bound for S because α is the **least** upper bound. Therefore, there is an $a_N \in S$ such that $\alpha - \epsilon < a_N$. Since $a_n \le a_{n+1}$ for all n, when $n > N$ we have $\alpha - \epsilon < a_N \le a_n \le \alpha$. This shows $\lim_{n\to\infty} a_n = \alpha$.

4. (5 pts.) One theorem from the past few weeks can be stated as $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$ there is an N such that $|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$ when $m, n > N$. Why is this true?

Answer: The hypothesis here amounts to: Given $\epsilon > 0$, there is an N such that the partial sums $s_n = \sum_{k=1}^{n} a_k$ satisfy $|s_n - s_m| < \epsilon$ when $n, m > N$. That says the partial sums are a Cauchy sequence. So the theorem says "$\sum_{n=1}^{\infty} a_n$ is convergent if and only if the partial sums are a Cauchy sequence". That is true because by definition $\sum_{n=1}^{\infty} a_n$ is convergent if and only if its sequence of partial sums converges, and a sequence converges if and only if it is a Cauchy sequence.

(b) (5 pts.) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: We need to show that the partial sums $s_n = \sum_{k=1}^{n} a_k$ form a Cauchy sequence. By the triangle inequality when $n > m$

$$|s_n - s_m| = |a_n + a_{n-1} + \cdots + a_{m+1}| \leq |a_n| + |a_{n-1}| + \cdots + |a_{m+1}| = |t_n - t_m|, \quad (1)$$

where $t_n = \sum_{k=1}^{n} |a_k|$ is a partial sum for $\sum_{n=1}^{\infty} |a_n|$. Since $\sum_{n=1}^{\infty} |a_n|$ converges, the t_n's are a Cauchy sequence, and (1) shows that this implies that the s_n's are a Cauchy sequence, too. So $\sum_{n=1}^{\infty} a_n$ converges.

5. (10 pts.) Prove the root test. This means prove that $\sum_{n=1}^{\infty} a_n$ converges if $\limsup_{n\to\infty} |a_n|^{1/n} = L < 1$, and diverges if $\limsup_{n\to\infty} |a_n|^{1/n} = L > 1$. You may use any theorems about limsup's from class and the homework without reproving them.

Proof: If $L < 1$, we can choose $\epsilon > 0$ so that $L + \epsilon < 1$. $\epsilon = (1 - L)/2$ is a good choice. Then we can use the theorem that, when $\limsup_{n\to\infty} |a_n|^{1/n} = L$, for every $\epsilon > 0$ there is an N such that $|a_n|^{1/n} < L + \epsilon$ for all $n > N$. So $|a_n| < (L + \epsilon)^n$ when $n > N$. Thus

$$\sum_{k=1}^{n} |a_k| \leq \sum_{k=1}^{N} |a_k| + \sum_{k=N+1}^{n} (L + \epsilon)^k \leq \sum_{k=1}^{N} |a_k| + \frac{1}{1 - L - \epsilon}$$

by the formula for the sum of a geometric series $\sum_{k=0}^{\infty} (1 - L - \epsilon)^{-k}$. This shows that the partial sums of $\sum_{n=1}^{\infty} |a_n|$ are bounded. So $\sum_{n=1}^{\infty} |a_n|$ converges. So $\sum_{n=1}^{\infty} a_n$ converges by problem 4(b).

If $L > 1$, we can choose $\epsilon > 0$ so that $L - \epsilon > 1$. $\epsilon = (L - 1)/2$ is a good choice. Then we can use the theorem that, when $\limsup_{n\to\infty} |a_n|^{1/n} = L$, for every $\epsilon > 0$ $|a_n|^{1/n} > L - \epsilon > 1$ infinitely often. So $|a_n| > 1$ infinitely often. So $\sum_{n=1}^{\infty} a_n$ does not converge.