1. Give the meaning of (= define) the following. In each case $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Be precise: no partial credit here.

(a) $\{a_n\}_{n=1}^{\infty}$ is bounded below.

**Definition:** $\{a_n\}_{n=1}^{\infty}$ is bounded below if and only if there is a $B \in \mathbb{R}$ such that $a_n \geq B$ for all $n$.

(b) $\lim_{n\to\infty} a_n = L \in \mathbb{R}$

**Definition:** $\lim_{n\to\infty} a_n = L \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n > N$.

(c) $\lim_{n\to\infty} a_n = +\infty$

**Definition:** $\lim_{n\to\infty} a_n = +\infty$ if and only if for every $M < \infty$ there is an $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$.

(d) $\{a_n\}_{n=1}^{\infty}$ is **not** a Cauchy sequence.

**Definition:** $\{a_n\}_{n=1}^{\infty}$ is **not** a Cauchy sequence if and only if for some $\epsilon_0 > 0$ there is **no** $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon_0$ for all $n, m > N$.

(e) $\limsup a_n$, assuming that $\{a_n\}_{n=1}^{\infty}$ is bounded above.

**Definition:** $\limsup a_n = \lim_{n\to\infty} b_n$ where $b_n = \sup\{a_k : k \geq n\}$. [Note that since $\{a_n\}_{n=1}^{\infty}$ is bounded above, $b_n < \infty$. If $\{a_n\}_{n=1}^{\infty}$ is not bounded **below**, then $\limsup a_n = \lim_{n\to\infty} b_n = -\infty$. This remark is not part of the definition.]

**Discussion:** I have written out the definitions in full logical style (with "if and only if"'s , lots of "such that"'s, and quantifiers in red) because you need to learn how to write like that. With the quantifiers incorrect or missing, these definitions degenerate into nonsense. In scoring the exam I wrote "OK" for answers that gave something equivalent to the property being defined, but were not what should be used as a definition. For instance, quite a few people said that a definition for (d) was $\{a_n\}_{n=1}^{\infty}$ is not convergent. That is logically equivalent to not being a Cauchy sequence, but, if you make it the definition, you end up defining Cauchy sequences as convergent sequences – which is not very useful.

2. For each sequence below find $\lim a_n$ if it is convergent, and both $\liminf a_n$ and $\limsup a_n$ if it is not. No proofs are necessary.

(a) $a_n = \frac{2^n+n^3}{(-2)^n+1}$

**Computation:** Dividing the numerator and denominator by $(-2)^n$ you find

$$a_n = \frac{(-1)^n + n^3(-2)^{-n}}{1 + (-2)^{-n}}$$

$(-2)^{-n}$ goes to zero as $n \to \infty$. $n^3(-2)^{-n}$ also goes to zero as $n \to \infty$. If you were asked to prove that you could do it by the ratio test, but you were not asked to prove it. Hence for $n$ even, i.e. for $n = 2m$, $m \in \mathbb{M}$, $a_n \to 1$ and for $n$ odd, i.e. for $n = 2m+1$, $a_n \to -1$. So $\limsup a_n = 1$ and $\liminf a_n = -1$.

(b) $a_n = (n + 2^n)^{2/n}$

**Computation:** Note that $a_n = 2^2(1 + n2^{-n})^{2/n}$. Since $n2^{-n} < 1$ for all $n$, the argument for problem 13 (b) in the homework shows $\lim_{n\to\infty}(1 + n2^{-n})^{2/n} = 1$. So $\lim_{n\to\infty} a_n = 2^2$.

(c) $a_n = 5^{1+(-1)^n}$

**Computation:** This was the easy one: when $n$ is even $a_n = 5^2 = 25$ and when $n$ is odd $a_n = 5^0 = 1$. So $\limsup a_n = 25$ and $\liminf a_n = 1$.

**Discussion:** In parts (a) and (b) the main trick was algebra: reducing the quotients in (a) to have a denominator equal to 1 plus a term that goes to zero, and factoring out 4 in (b) so that you were left with $(1 + b_n)^{2/n}$ where $b_n$ is bounded.

3. In this problem and the next you are asked for proofs of theorems from this course. Please write these carefully.

Assume that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers that is bounded below. If $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence, prove that $\{a_n\}_{n=1}^{\infty}$ is convergent.

Proof: The standard proof goes like this: Since $\{a_n\}_{n=1}^{\infty}$ is bounded below, there is a $B \in \mathbb{R}$ such that $a_n \geq B$ for all $n$. Hence, by the completeness axiom the set $S = \{a_n : n \in \mathbb{N}\}$ has a greatest lower bound $\beta$. For every $\epsilon > 0$ $\beta + \epsilon$ cannot be a lower bound for $S$, because $\beta$ is the greatest lower bound, so there is an element of $S$, $a_{n_0}$, satisfying $a_{n_0} < \beta + \epsilon$. Since $\beta$ is a lower bound for $S$, $\beta \leq a_n$ for all $n$ and, since $\{a_n\}_{n=1}^{\infty}$ is decreasing, $a_{n+1} \leq a_n$ for all $n$. Putting these inequalities together (using the transitivity of $\leq$) gives for $n > n_0$

$$\beta \leq a_n \leq a_{n_0} < \beta + \epsilon$$

Setting $N = n_0$ and using the definition of $\lim_{n\to\infty} a_n$ from problem 1(b), we have $\lim_{n\to\infty} a_n = \beta$.

4. Assume that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. Prove that, if $\limsup a_n = \liminf a_n$, then $\{a_n\}_{n=1}^{\infty}$ is convergent.

Proof: The standard proof goes like this: Using problem 1(e), $L = \limsup a_n = \lim_{n\to\infty} b_n$ where $b_n = \sup\{a_k : k \geq n\}$. So, for every $\epsilon > 0$ there is an $N_1$ such that $b_n < L + \epsilon$ for all $n > N_1$. Likewise, since we are assuming that $\limsup a_n = \liminf a_n$, we have $L = \liminf a_n = \lim_{n\to\infty} c_n$ where $c_n = \inf\{a_k : k \geq n\}$. So there is also an $N_2$ such that $c_n > L - \epsilon$ for all $n > N_2$. Finally, by the definitions of inf and sup, $c_n \leq a_n \leq b_n$. So putting these inequalities together (using the transitivity of $\leq$) gives for $n > N = \max\{N_1, N_2\}$

$$L - \epsilon < c_n \leq a_n \leq b_n < L + \epsilon.$$

So by the definition of $\lim_{n\to\infty} a_n$ from problem 1(b), we have $\lim_{n\to\infty} a_n = L$, proving that $\{a_n\}_{n=1}^{\infty}$ is convergent.

5. Use the limit theorems, $\lim_{n\to\infty} 1/n = 0$, and algebra to compute the following limits, and **prove** that your computations are correct.

(a) $\lim_{n\to\infty} \frac{3n+n^{-1}}{1+2n}$

**Proof:** This one is not too hard:

$$\lim_{n\to\infty} \frac{3n + n^{-1}}{1 + 2n} = \lim_{n\to\infty} \frac{3 + n^{-2}}{2 + n^{-1}} \text{ (algebra)}$$

$$= \text{(provided that all these limits exist)} \frac{3 + (\lim_{n\to\infty} n^{-1})^2}{2 + \lim_{n\to\infty} n^{-1}} = \frac{3 + 0^2}{2 + 0} = 3/2.$$

In that I used limit of the quotient is the quotient of the limits (which only works because the limit of the denominator is 2 ($\neq 0$)). Also in both the numerator and the denominator I used limit of the sum is the sum of the limits (with $\lim_{n\to\infty} 3 = 3$ and $\lim_{n\to\infty} 2 = 2$) and finally in the numerator I used limit of the product is the product of the limits to get $\lim_{n\to\infty} n^{-2} = (\lim_{n\to\infty} n^{-1})^2$.

(b) $\lim_{n\to\infty} \sqrt{a_n^2 + n^2} - n$, when $\lim_{n\to\infty} a_n = A \in \mathbb{R}$.

**Proof:** This one is shorter but less obvious.

$$\lim_{n\to\infty} \sqrt{a_n^2 + n^2} - n = \lim_{n\to\infty} \frac{a_n^2}{\sqrt{a_n^2 + n^2} + n} \text{ (algebra)}.$$

At this point you have to notice that

$$0 < \frac{a_n^2}{\sqrt{a_n^2 + n^2} + n} < \frac{a_n^2}{n}$$

By limit of the product equals product of the limits

$$\lim_{n\to\infty} \frac{a_n^2}{n} = (\lim_{n\to\infty} a_n)^2 (\lim_{n\to\infty} n^{-1}) = A \cdot A \cdot 0 = 0.$$

So by the Squeeze Theorem $\lim_{n\to\infty} \sqrt{a_n^2 + n^2} - n = 0$. I forgot to say that you could use the Squeeze Theorem, but I hope that you used it anyway.