Midterm 2 : 131A

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May 17 2019

This test totals 40 points and you get 50 minutes to do it. Answer the questions in the spaces provided on the question sheets. Always show work unless the question says otherwise. Good luck!

Name : ______________________

ID number : ______________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Assume $s_1 = \sqrt{2}$. Define $s_{n+1} = \sqrt{2s_n}$ for $n \geq 1$ recursively. Find $\lim s_n$. Reason rigourously at every step! [Hint: Show s_n is monotonically increasing and 2 is an upper bound]

- Note by def $s_n \geq 0 \quad \forall\, n \in \mathbb{N}$
- $s_1 = \sqrt{2} < 2$
- Assume by induction that $s_n < 2$
- Show $s_{n+1} \geq s_n$ and $s_{n+1} < 2$

Pf. $s_{n+1} = \sqrt{2 s_n}$

$s_{n+1}^2 = 2 s_n < 2 \times 2 = 4$ as $s_n < 2$

$\Rightarrow s_{n+1} < \sqrt{4} = 2$ (+ve sq root)

Also $s_{n+1}^2 = 2 s_n > s_n^2$ as $s_n < 2$

$\Rightarrow s_{n+1} > s_n$ (+ve sq roots)

So $s_n \uparrow$, upper bound $= 2$ (by induction)

So $\lim s_n = s \in \mathbb{R}$ [monotonic $\uparrow$, upperbd seq conv]

$\Rightarrow \lim_n s_{n+1} = \lim_n s_n = s$

$\Rightarrow \lim_n \sqrt{2 s_n} = s$

$\Rightarrow \sqrt{2s} = s$

$\Rightarrow 2s = s^2 \Rightarrow s(s-2) = 0 \Rightarrow s = 2$ or $s = 0$

but $s_n \uparrow$ and $s_n \geq \sqrt{2}$ so $s = \sup(s_n) > \sqrt{2} \Rightarrow \boxed{s = 2}$ $(s \neq 0)$

2. (10 points) Let $s_n = (-1)^n \left(\frac{n^2-1}{n^2+1}\right)$ for $n \geq 1$. Find $\limsup s_n$ and $\liminf s_n$. Justify your answers rigourously.

$$|s_n| = \left|\frac{n^2-1}{n^2+1}\right| \quad \text{as} \quad n \geq 1 \Rightarrow \begin{array}{l} n^2-1 \geq 0 \\ n^2+1 > 0 \end{array}$$

$n > m > 0$

$$\Rightarrow n^2+1 > m^2+1 \Rightarrow 0 < \frac{2}{n^2+1} < \frac{2}{m^2+1}$$

$$\Rightarrow 0 > \frac{-2}{n^2+1} > \frac{-2}{m^2+1}$$

$$\Rightarrow 1 - \frac{2}{n^2+1} > 1 - \frac{2}{m^2+1}$$

$$\Rightarrow \frac{n^2-1}{n^2+1} > \frac{m^2-1}{m^2+1}$$

so $|s_n| \uparrow$ seq, $|s_n| = \begin{cases} -s_n & n \text{ odd} \\ s_n & n \text{ even} \end{cases}$

$\Rightarrow 0 < -s_1 < s_2 < -s_3 < s_4 \ldots$

So

$\ldots s_5 < s_3 < s_1 < 0 < s_2 < s_4 < s_6 \ldots$

$\uparrow$

$(0 < -s_1 < -s_3 < -s_5 \ldots$

$\Rightarrow 0 > s_1 > s_3 > s_5 \ldots)$

$u_i = \inf \{ s_{i+1}, s_{i+2} \ldots \}$

$= \inf \{ s_n \mid n \text{ odd}, n > i \} =$ ~~$\lim s_n$ n odd $= \lim (-1)\left(\frac{n^2-1}{n^2+1}\right)$~~

$$= \lim_{k\to\infty} (-1)^{2k+1} \left(\frac{(2k+1)^2 - 1}{(2k+1)^2 + 1} \right)$$

$$= \lim_{k\to\infty} -1 \left(\frac{1 - \frac{1}{(2k+1)^2}}{1 + \frac{1}{(2k+1)^2}} \right)$$

$$= -1$$

So $u_i = -1$

$$\Rightarrow \boxed{\lim u_i = -1 = \liminf s_n}$$

$$v_i = \sup \{ s_{i+1}, s_{i+2}, \dots \}$$

$$= \sup \left\{ s_n \;\middle|\; \begin{matrix} n \text{ even} \\ n > i \end{matrix} \right\}$$

$$= \lim_{k\to\infty} (-1)^{2k} \left(\frac{(2k)^2 - 1}{(2k)^2 + 1} \right)$$

$$= \lim_{k\to\infty} (+1) \left(\frac{1 - \frac{1}{(2k)^2}}{1 + \frac{1}{(2k)^2}} \right)$$

$$= 1$$

$$\Rightarrow \boxed{\lim v_i = 1 = \limsup s_n}$$

3. $\overset{\text{show}}{\boxed{T \text{ closed}}}$ if $T = \{1, 2, .. 10\}$

Let (t_n) be a seq where $t_n \in T$

Suppose $\lim_{n \to \infty} t_n = t$

so given $\varepsilon = 0.01 > 0$,

there is $N \in \mathbb{N}$, so that for all $n > N$,

$$|t_n - t| < 0.01$$

Since $t_n \in \{1, 2, \ldots, 10\}$ for all n

let for some $n_* > N$, $t_{n_*} = i$ $\quad i \in \{1, 2, .., 10\}$

so $|t_{n_*} - t| < 0.01$

If for some $n > N$, $t_n = j$ $\quad j \neq i$, $j \in \{1, 2, .. 10\}$

then $|t_n - t| < 0.01$

but

$$|j - i| = |t_n - t_{n_*}| \leq |t_n - t| + |t - t_{n_*}|$$

$$< 0.01 + 0.01 = 0.02$$

but $j, i \in \{1, 2 .., 10\}$ so $|j - i| \geq 1$

this shows, $\overset{\text{contradiction}}{\cancel{\text{so}}}$ that for <u>all</u> $n > N$, $t_n = t_{n_*} = i$

So $(t_n) = \{t_1, .., t_N, i, i, i, i \ldots\}$

So $\lim t_n = i \in T$ $\quad \Rightarrow T$ closed.
$= t$

4. (10 points) Let (s_n) be a sequence so that $|s_{n+1} - s_n| \leq \frac{|s_n - s_{n-1}|}{2}$ for all $n \geq 2$. Suppose further that $s_1 = 1$ and $s_2 = 2$. Show that (s_n) is Cauchy. Reason rigourously at every step!

$s_1 = 1 \qquad s_2 = 2 \qquad |s_2 - s_1| = 1$

$$|s_3 - s_2| \leq \frac{1}{2}$$

$$|s_4 - s_3| \leq \frac{|s_3 - s_2|}{2} \leq \frac{1}{2^2}$$

$$|s_5 - s_4| \leq \frac{|s_4 - s_3|}{2} \leq \frac{1}{2^3}$$

Assume $|s_{n+1} - s_n| \leq \frac{1}{2^{n-1}}$ for some $n \geq 2$

then $|s_{n+2} - s_{n+1}| \leq \frac{|s_{n+1} - s_n|}{2} \leq \frac{1}{2}\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n}$

Thus by induction $|s_{k+1} - s_k| \leq \frac{1}{2^{k-1}}$ for all $k \geq 2$

given $\varepsilon > 0$, pick $N^* \in \mathbb{N}$ so that

$$\frac{1}{2^{N^*}} < \varepsilon$$

set $N = N^* + 2$

so $0 < \frac{1}{2^{N-2}} = \frac{1}{2^{N^*}} < \varepsilon$

for $m \geq n > N$

$$|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}|$$

$$\cdots + |s_{n+1} - s_n|$$

$$\leq \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}} \left(\frac{1}{2^{m-n-1}} + \frac{1}{2^{m-n-2}} \quad + 1 \right)$$

$$= \frac{1}{2^{n-1}} \left(\frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}} \right)$$

$$= \frac{1}{2^{n-2}} \left(1 - \left(\frac{1}{2}\right)^{m-n} \right)$$

$m - n \geq 0$

$$< \frac{1}{2^{N-2}} (1)$$

$$< \varepsilon$$

so (s_n) Cauchy