Midterm 1 : 131A

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April 26 2019

This test totals 50 points and you get 50 minutes to do it. Answer the questions in the spaces provided on the question sheets. Always show work unless the question says otherwise. Good luck!

Name : ____________________

ID number : ____________________

Question	Points	Score
1	10	
2	10	
3	5	
4	15	
5	10	
Total:	50	

1. (10 points) Assume $n \geq 2$. From first principles (ϵ-definition) and reasoning rigourously at every step, find

$$\lim_{n\to\infty}\left(\frac{1}{n+1}-\frac{2}{n-1}\right).$$

$L = 0$

given $\varepsilon > 0$, find $N \in \mathbb{N}$ so that

$$\left|\frac{1}{n+1} - \frac{2}{n-1} - 0\right| < \varepsilon \quad \text{for all } n > N$$

Since $\left|\frac{1}{n+1} - \frac{2}{n-1}\right| \leq \left|\frac{1}{n+1}\right| + \left|\frac{2}{n-1}\right| = \frac{1}{n+1} + \frac{2}{n-1}$

by Δ ineq

enough to ensure

$$\frac{1}{n+1} + \frac{2}{n-1} < \varepsilon \quad \text{for all } n > N$$

By arch property, given $\varepsilon > 0$, find $\tilde{N} \in \mathbb{N}$ so that

$$\frac{1}{\tilde{N}} < \frac{\varepsilon}{4}$$

Set $N = \tilde{N} + 5$

so $n > N \Rightarrow$ $n - 1 > N - 1 = \tilde{N} + 4 > \tilde{N}$

$n + 1 > N + 1 = \tilde{N} + 6 > \tilde{N}$

In any case, for all $n > N$

$$\left|\frac{1}{n+1} - \frac{2}{n-1}\right| \leq \left|\frac{1}{n+1}\right| + \left|\frac{2}{n-1}\right| < \frac{1}{\tilde{N}} + \frac{2}{\tilde{N}} < \frac{\varepsilon}{4} + \frac{2\varepsilon}{4}$$

$$< \varepsilon.$$

2. Let A and B be non-empty bounded subsets[1] of $\mathbb{R}$ such that $A \cap B \neq \emptyset$

(a) (4 points) Show $\sup(A \cup B)$ exists and $\sup(A \cup B) = \max(\sup(A), \sup(B))$.

~~[illegible]~~

A, B non empty $\Rightarrow$ $A \cup B$ non empty.

Let $M = \max(\sup(A), \sup(B))$, in particular $M \geq a$ for all $a \in A$, $M \geq b$ for all $b \in B$

$\Rightarrow M$ upper bound for $A \cup B$

$\Rightarrow$ $\sup(A \cup B)$ exists by completeness axiom and $M \geq \sup(A \cup B) = L$ call it L

Since $L = \sup(A \cup B)$, $L \geq a$ for all $a \in A$, $L \geq b$ for all $b \in B$ $\Rightarrow L \geq \sup(A), \sup(B)$

$\Rightarrow L \geq \max(\sup(A), \sup(B)) = M$.

$\Rightarrow \boxed{L = M}$

(b) (4 points) Show that $\sup(A \cap B)$ exists and $\sup(A \cap B) \leq \min(\sup(A), \sup(B))$

$A \cap B \neq \emptyset$ given. Let $\sup(A) = \alpha$, $\sup(B) = \beta \in \mathbb{R}$

so $\alpha \geq a$ for all $a \in A$

$\beta \geq b$ for all $b \in B$

$\Rightarrow \min(\alpha, \beta) \geq x$, for all $x \in A \cap B$

$\Rightarrow \min(\alpha, \beta)$ is an upper bound for $A \cap B$

and so $\sup(A \cap B)$ exists by completeness axiom and

$\min(\alpha, \beta) \geq \sup(A \cap B)$

(c) (2 points) Give an example of bounded non-empty sets A, B so that $\sup(A \cap B)$ exists and $\sup(A \cap B) < \min(\sup(A), \sup(B))$.

$A = (1, 2)$ $\quad$ $\sup A = 2$

$B = (1, 1.5) \cup \{2\}$ $\quad$ $\sup B = 2$ $\quad$ min = 2

$A \cap B = (1, 1.5)$ $\quad$ $\sup A \cap B = 1.5$

$1.5 < 2$

[1]Recall $A \cup B$ is the set of all x which is in A or B or both. Similarly $A \cap B$ is x which is in both A and B simultaneously.

3. (5 points) Let (s_n) be a sequence such that $\lim_{n\to\infty} s_n = -\infty$. Prove from first principles that $\lim_{n\to\infty} s_n^2 = +\infty$.

Given any $\tilde{M} < 0$, there is $N_{\tilde{M}} \in \mathbb{N}$ so that $s_n < \tilde{M}$ for all $n > N_{\tilde{M}}$ ☆ as $\lim s_n = -\infty$

Given $M > 0$, choose $M_1 > \max(1, M)$ (by arch prop)

so $M_1^2 > M_1 > 1, M > 0$

$\boxed{\text{So } \tilde{M} = -M_1^2}$

By ☆ find $N_{\tilde{M}}$.

so $s_n < -M_1^2$ for all $n > N_{\tilde{M}}$

so $-s_n > M_1^2 > M$ for all $n > N_{\tilde{M}}$
> 0

$\Rightarrow s_n^2 > M_1^4 > M_1^2 > M > 0$

for all $n > \tilde{N}_M$

$\Rightarrow \lim_{n\to\infty} s_n^2 = M$

4. Let (s_n) be a sequence so that each $s_n > 0$. Suppose you know that $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \frac{1}{3}$.

(a) (5 points) Show that there is an $N \in \mathbb{N}$ so that $s_{n+1} < \frac{1}{2}s_n$ for all $n \geq N$

Given $\varepsilon > 0$, there is $N_1 \in \mathbb{N}$ so that

$$\left| \frac{s_{n+1}}{s_n} - \frac{1}{3} \right| < \varepsilon$$

(ie) $\frac{s_{n+1}}{s_n} \in \left(\frac{1}{3} - \varepsilon, \frac{1}{3} + \varepsilon \right)$ for all $n > N_1$)

choose $\varepsilon < \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\Rightarrow \frac{s_{n+1}}{s_n} \in \left(\frac{1}{6}, \frac{1}{2} \right)$ for all $n > N_1$

$\Rightarrow \frac{s_{n+1}}{s_n} < \frac{1}{2}$ for all $n > N_1$

$N = N_1 + 1$

$\Rightarrow \frac{s_{n+1}}{s_n} < \frac{1}{2}$ for all $n \geq N$,

(b) (5 points) Show using the above part that $s_n < \left(\frac{1}{2^{n-N}}\right) s_N$ for all $n > N$.

$$s_{N+1} < \frac{1}{2} s_N$$

$$s_{N+2} < \frac{1}{2} s_{N+1} < \frac{1}{2}\left(\frac{1}{2}\right) s_N = \frac{s_N}{2^{N+2-N}}$$

induction, assume hyp

$$s_n < \frac{1}{2^{n-N}} s_N \quad \text{for all } n > M$$

Then

$$s_{n+1} < \frac{1}{2} s_n < \frac{1}{2 \cdot 2^{n-N}} s_N$$

$$= \frac{1}{2^{n+1-N}} s_N$$

Hence by induction, $s_n < \left(\frac{1}{2^{n-N}}\right) s_N$ for all $n > N$

(c) (5 points) Hence find $\lim_{n\to\infty} s_n$ (you can use squeeze lemma).

~~Define seq $s''_n = 0$~~

Define seq $$s'_n = \frac{s_N}{2^{n-N}} \quad \text{for all } n > N$$

$$s'_n = \max(s_1, \dots s_N) + 1 \quad \text{for } n \leq N$$

so $s''_n \leq s_n \leq s'_n$ for all $n \in \mathbb{N}$

$$\lim_{n\to\infty} s'_n = \boxed{2^N s_N} \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

constant

$\lim s''_n = 0$

By squeeze lemma

$\lim s_n = 0.$

3) Let $r = 2$. Find $s_n \in \mathbb{Q}$, $s_n < s_{n+1}$ so that $s_n \to 2$

Pf. $s_n = 2 - \frac{1}{n}$ $\quad n \geq 1$

Claim: $s_n < s_{n+1}$ for all $n \geq 1$

because $n+1 > n$,

$$\frac{1}{n+1} < \frac{1}{n}$$

$$\Rightarrow \frac{-1}{n+1} > \frac{-1}{n}$$

$$\Rightarrow 2 - \frac{1}{n+1} > 2 - \frac{1}{n}$$

$$\Rightarrow s_{n+1} > s_n$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2 - \frac{1}{n} = \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{1}{n}$$

$$= 2 - 0$$

$$= \underline{2}$$

*****************SCRATCH WORK ********************************