Problem 1. Let (s_n), (t_n) be sequences of real numbers.

1. State the definition of the following statements: "(s_n) is convergent" and "(t_n) is divergent."
2. Give (without proof) an example of two divergent sequences (s_n) and (t_n) such that $(s_n + t_n)$ is convergent.
3. Yes or no (without proof): there are no two convergent sequences (s_n) and (t_n) such that $(s_n + t_n)$ is divergent.

(10 + 5 +5 points.)

Answer: 1. There exists $s, t \in \mathbb{R}$ such that (s_n) approachs s as $n \to \infty$

we have :

(s_n) convergent. For all $\varepsilon > 0$, $\exists n_1 \geq m$ s.t. $|s_n - s| < \varepsilon$ $\forall n \geq n_1$ 5/5

(t_n) divergent: There exists $\varepsilon > 0$, $\forall n_2 \geq m$ s.t. $\exists n \geq n_2$ where $|t_n - t| \geq \varepsilon, \forall t$

where m is the starting index, $\varepsilon \in \mathbb{R}$, $n, n_1, n_2, m \in \mathbb{Z}$

2/5.

2.

Let $s_n = (-1)^n$, $t_n = (-1)^{n+1}$ where $n \in \mathbb{N}$

5/5.

3. Yes.

5/5.

Problem 2. Let (s_n) be a sequence of real numbers. Show that

$$s_n \to 0 \quad \Longleftrightarrow \quad s_n \cdot s_n \to 0.$$

(20 points.)

Answer:

$\Rightarrow)$ By theorem 9.2, 9.3; Suppose $s_n \to 0$

$\lim(s_n \cdot s_n) = (\lim s_n) \cdot (\lim s_n) = 0 \cdot 0 = 0$ ✓

$\therefore \; s_n \cdot s_n \to 0$

$\Leftarrow)$ Let $\varepsilon > 0$ be given.

Since $s_n \cdot s_n = s_n^2 \to 0$, $\exists N_1 \geq m$ s.t. $|s_n^2 - 0| = |s_n^2| < \varepsilon^2 \quad \forall n \geq N_1$

Now:

$|s_n - 0| = |s_n| = \sqrt{|s_n|^2} = \sqrt{|s_n^2|} < \sqrt{\varepsilon^2} = |\varepsilon| = \varepsilon \quad \forall n \geq N_1$

$\therefore \; s_n \to 0$ ✓

$\therefore \; s_n \to 0 \iff s_n \cdot s_n \to 0$

Problem 3.

Let S be a nonempty set of positive real numbers with $\inf S > 0$, and let

$$S^{-1} := \{s^{-1} : s \in S\}.$$

Show that

$$\sup S^{-1} = (\inf S)^{-1}.$$

(20 points.)

Answer: From the question, we know that inf S exists

Let $s_0 = \inf(S) > 0 \Rightarrow s > 0 \quad \forall s \in S$

By def., ① $0 < s_0 \leq s \quad \forall s \in S$.

② if $\exists s_1$ s.t. $s_1 \leq s \quad \forall s \in S$, then $s_1 \leq s_0$

$\Leftrightarrow$ ① $0 < s^{-1} \leq s_0^{-1} \quad \forall s \in S$ (since $s_0 > 0$, $s > 0$ and $s_0 \leq s$ and property of ordered field)

② if $\exists s_2$ s.t. $s^{-1} \leq s_2 \quad \forall s \in S$, then let $s_2 = s_1^{-1} \Rightarrow s^{-1} \leq s_1^{-1}$
$\Rightarrow s_1 \leq s$

Therefore, $s_1 \leq s_0$ and so $s_0^{-1} \leq s_1^{-1} = s_2$

$\therefore$ The above definition ① says that s^{-1} is bounded above, so there exists a $\sup(S^{-1})$

and ② that s_0^{-1} is indeed a $\sup(S^{-1})$

$\therefore \sup(S^{-1}) = [\inf(S)]^{-1}$

✓

Problem 4.

1. State the Triangle Inequality.
2. Using part (1.), show by induction on n that for all integers $n \geq 1$ and $a_1, \ldots, a_n \in \mathbb{R}$:

$$|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|.$$

(10 + 10 points.)

Answer:

1. For $a, b \in R$

$|a+b| \leq |a| + |b|$ 10/10

2. Let $P(n)$ be a statement where $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$

Base Case: $P(1)$: $|a_1| \leq |a_1|$, which is trivially true.

Now assume $P(n)$ is true, we show $P(n+1)$ is true.

$P(n+1)$: $|a_1 + \cdots + a_n + a_{n+1}| \leq |a_1 + \cdots + a_n| + |a_{n+1}|$ by (1)

$\leq |a_1| + \cdots + |a_n| + |a_{n+1}|$ by $P(n)$

$\therefore$ By induction, $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$ $\forall n \geq 1$

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Problem 5. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and $a \in \mathbb{R}$ such that $a_n \to a$. Consider the sequence $(b_n)_{n\geq 1}$ given by

$$b_n = \begin{cases} 27 & \text{if } n < 1000 \\ a_n & \text{if } n \geq 1000. \end{cases}$$

Using only the definition of convergence of sequences, show that $b_n \to a$.

(20 points.)

Answer:

Let $\varepsilon > 0$ be given

$n_1 \geq 1000$?

Since $a_n \to a$, $\exists n_1$ s.t. $|a_n - a| < \varepsilon \quad \forall n \geq n_1 \geq 1000$

Now, as $n \to \infty$, we take $n_0 \geq n_1 \geq 1000$, or $b_n = a_n$

Therefore, $|b_n - a| = |a_n - a| < \varepsilon \quad \forall n \geq n_0$

$\therefore b_n \to a$ as $n \to \infty$

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