20

1. Let $(s_n)$ be a sequence of real numbers.

a) Define $\limsup s_n$ and $\liminf s_n$ (which may be $\infty$ or $-\infty$).

$\limsup s_n$ is $\lim_{N \to \infty} \sup\{s_n : n > N\}$

$\liminf s_n$ is $\lim_{N \to \infty} \inf\{s_n : n > N\}$

b) Assume the sequence $(s_n)$ is bounded: for some real number $M \geq 0$, $|s_n| \leq M$ for all $n$, and assume $\liminf s_n = \limsup s_n = \alpha$ (which is a real number since $(s_n)$ is bounded).

Prove the sequence $(s_n)$ converges to $\alpha$.

Since $\limsup s_n = \alpha$, for every $\varepsilon > 0$, $\exists N_1$ such that $n > N_1$ implies $s_n < \alpha + \varepsilon$.

Since $\limsup s_n$ is also $\alpha$, $\exists N_2$ such that $n > N_2$ implies $s_n > \alpha - \varepsilon$

Let $N = \max\{N_1, N_2\}$, $n > N$ implies $\alpha - \varepsilon < s_n < \alpha + \varepsilon$

$|s_n - \alpha| < \varepsilon$

Since for every $\varepsilon > 0$, $\exists N$ such that $n > N$ implies $|s_n - \alpha| < \varepsilon$

$\Rightarrow \lim_{n \to \infty} s_n = \alpha$

The sequence $(s_n)$ converges to $\alpha$ (proven)

19

2. Do the following series converge or diverge? Give a reason for your answer.

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

Since $\sum \frac{1}{n^p}$ converges if and only if $p>1$, this series diverges

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$.

Since $\lim_{n\to\infty} \frac{(-1)}{n^{\frac{1}{3}}} = 0$, the series is an alternating series, $\frac{1}{\sqrt[3]{n}} \downarrow$ $-1$

the series converges.

c) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Ratio $\lim_{n\to\infty} \left| \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{2n^2} \right| = \lim_{n\to\infty} \left| \frac{n^2+2n+1}{2n^2} \right| = \frac{1}{2} < 1$

~~The~~ Since $\limsup |a_{n+1}/a_n| = \lim |a_{n+1}/a_n| = \frac{1}{2} < 1$,

by ratio test, the series converges.

d) $\sum_{n=1}^{\infty} \frac{(10)^n}{n!}$.

Ratio $\lim_{n\to\infty} \left| \frac{10^{n+1}}{(n+1)!} \Big/ \frac{10^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{10}{n+1} \right| = 0$

Since $\limsup |a_{n+1}/a_n| = \lim |a_{n+1}/a_n| = 0 < 1$

By ratio test, the series converges

3. Let $(a_n)$ be a sequence of real numbers such that $\limsup |a_n|^{1/n} = \alpha$, where $0 < \alpha < \infty$. Prove that for a real number $x$ the series

$$\sum_{n=1}^{\infty} a_n x^n$$

20

converges if $|x| < 1/\alpha$ and diverges if $|x| > 1/\alpha$. (Nothing can be said when $|x| = 1/\alpha$.)

Hint: Root test.

Since $x$ is a real number, consider a constant sequence $S_n$ with all elements being $x^n$,

Thus, $\lim |S_n|^{\frac{1}{n}} = |x^n|^{\frac{1}{n}} = |x|$

$\limsup |a_n x^n|^{\frac{1}{n}} = \limsup (|a_n| |x^n|)^{\frac{1}{n}}$

$= \limsup (|a_n|^{\frac{1}{n}} |x^n|^{\frac{1}{n}})$

Since for a sequence $S_n$ that converges to $S$, $t_n$ is any sequence, $\limsup S_n t_n = S \cdot \limsup t_n$

$\limsup (|a_n|^{\frac{1}{n}})(|x^n|^{\frac{1}{n}}) = |x| \cdot \limsup |a_n|^{\frac{1}{n}}$

$= |x| \cdot \alpha$

$= \alpha |x|$

Hence if $|x| < \frac{1}{\alpha}$, $\limsup |a_n x^n|^{\frac{1}{n}} < 1$, by root test, $\sum_{n=1}^{\infty} a_n x^n$ converges

If $|x| > \frac{1}{\alpha}$, by root test, the series diverges

The test gives no information when $|x| = \frac{1}{\alpha}$ since $\limsup |a_n x^n|^{\frac{1}{n}} = 1$

$(\alpha \cdot \frac{1}{\alpha} = 1)$

16

4. a) Prove that the function $f(x) = \sqrt{x}$ is continuous on $[1, \infty)$.

let $x_0$ be any point in dom $f$, $|\sqrt{x} - \sqrt{x_0}| = \left|\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\right| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{|x - x_0|}{\sqrt{x}}$

for any $\varepsilon > 0$, let $|x - x_0| < \delta = \varepsilon\sqrt{x_0}$ which pt fixed

$|\sqrt{x} - \sqrt{x_0}| < \frac{|x - x_0|}{\sqrt{x}} < \frac{\delta}{\sqrt{x}} = \frac{\varepsilon\sqrt{x}}{\sqrt{x}} = \varepsilon$ $\sqrt{x_0} > 0$. $-2$

Hence for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $x \in$ domf and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon \Rightarrow f(x) = \sqrt{x}$ is continuous on $[1, \infty)$

b) For $x$ real define $[x]$ to be the greatest integer less than or equal to $x$. Prove $f(x) = [x]$ is continuous at $x$ if and only if $x$ is not an integer.

Let $x_0$ be an integer.

Let $\varepsilon = \frac{1}{2}$, let $x < x_0$

Since $|f(x) - f(x_0)| \geq 1$, there is no $\delta$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \frac{1}{2}$. Gap. Thus $x_0$ is not an integer. $-2$ left limit right limit diff is "1"

If $x_0$ is not an integer, for each $\varepsilon > 0$

let $\delta = \frac{1}{2} \times \min\{|\lceil x \rceil - x_0|, |\lfloor x \rfloor - x_0|\}$

Thus, when $|x - x_0| < \delta$, $f(x) = [x] = f(x_0) = [x_0] = \lfloor x_0 \rfloor$

$|f(x) - f(x_0)| = 0 < \varepsilon$

Thus, $f(x) = [x]$ is continuous at $x$ if and only if $x$ is not an integer

5. For all real $x$ define Define the function $F(x)$ by:

$F(x) = 0$ if $x$ ~~is not rational~~; $x \in \mathbb{R} \setminus \mathbb{Q}$

$F(x) = 1/m$ if ~~$x = n/m$~~ is ~~rational and the integers~~ $n$ ~~and~~ $m$ ~~have no common factor~~ (except 1 and −1). if $x \in \mathbb{Q} \neq 0$, $x = \frac{n}{m}$, $(n, m) = 1$, $m > 0$, $F(x) = 0$, $x = 0$

Prove $F$ is continuous at $x$ if and only if $x$ is not rational.

8.

If $x_0$ is rational $x_0 = \frac{n}{m}$, let $\varepsilon = \frac{1}{2m}$

for each $\delta > 0$, there exists infinitely many irrational numbers $x$ in $[x_0, x_0 + \delta]$ and that $|x - x_0| < \delta$

$|f(x) - f(x_0)| = |0 - \frac{1}{m}| = \frac{1}{m} > \frac{1}{2m} = \varepsilon$ ✓

Thus, for $\varepsilon = \frac{1}{2m}$, there is no $\delta$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$

Hence $x$ cannot be rational

If $x_0$ is not rational, $f(x_0) = 0$. Since rationals are countable, let $\delta$ be the absolute value of the difference between $x_0$ and its closest rational,

if $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| = 0 - 0 = 0 < \varepsilon > 0$

Hence, for each $\varepsilon > 0$, $\exists \delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. F is continuous at $x$ if and only if $x$ is not rational.