1. a) State the Principle of Mathematical Induction.

The Principle od Mathematical Induction state that if

② when a statement is true for a natural number $n=k$, then it will also be true to its successor $n=k+1$

① the statement is true for an initial $n=N$, N is a natural number

then the statement is true for any natural number $n \geq N$

b) Prove using induction that for all positive integers n,

$$1 + 3 + 5 + ... + (2n - 1) = n^2.$$

Proof:

Let P_n denote $1+3+5+\dots+(2n-1)=n^2$

P_1: $1=1^2$ true

Assume P_k is true for $k \geq 1$, $k \in \mathbb{N}$

$\Rightarrow 1+3+\dots+(2k-1)=k^2$

$1+3+\dots+(2k-1)+[2(k+1)-1] = k^2+[2(k+1)-1]$

$1+3+\dots+(2k-1)+(2k+1) = k^2+2k+1 = (k+1)^2$

$\Rightarrow P(k+1)$ is true

Since P_1 is true, P_k is true $\Rightarrow P_{(k+1)}$ is true, By the Principle od Mathematical induction, $P(n)$ is true for all $n \geq 1$, $n \in \mathbb{N}$.

2. a) Clearly state the Archimedean Property of real numbers.

If $a>0$, $b>0$, $a,b \in \mathbb{R}$, there exists some positive integer n such that $na > b$.

b) Prove that whenever a and b are real numbers and $a < b$ there exists a rational number r such that $a < r < b$.

Proof:

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Since $a<b$, $b-a>0$, by the Archimedean Property of real numbers, $\exists$ positive integer n such that $n(b-a)>1$ since $1>0$

$nb-na>1$

$nb>na+1$

$nb>na+1>na$

$b>a+\frac{1}{n}>a$

Since $n \in \mathbb{Z}$, $1 \in \mathbb{N}$, $\frac{1}{n} \in \mathbb{R}$

since $a \in \mathbb{R}$, $\frac{1}{n} \in \mathbb{R}$, $a+\frac{1}{n} \in \mathbb{R}$

let $r = a+\frac{1}{n} \in \mathbb{R}$, we have

$a<r<b$ (proven)

3. (a)For a sequence (s_n) of real numbers, and for real number s, give the definition of

$\lim_{n\to\infty} s_n = s$. For every $\varepsilon > 0$, there exists N such that for every $n > N$, $|s_n - s| < \varepsilon$. Then $\lim_{n\to\infty} s_n = s$

Let $s_n = \frac{1}{1+5n^{1/3}}$. For any $\epsilon > 0$ find a real number N so that if $n > N$ then $0 < s_n < \epsilon$.

For $n > 0, \varepsilon > 0$, $\frac{1}{1+5n^{\frac{1}{3}}} < \varepsilon$ $\quad (1+5n^{\frac{1}{3}} > 0)$

$$\frac{1}{\varepsilon} < 1+5n^{\frac{1}{3}}$$

$$\frac{1}{\varepsilon} - 1 < 5n^{\frac{1}{3}}$$

$$\frac{1-\varepsilon}{5\varepsilon} < n^{\frac{1}{3}}$$

$$n > \left(\frac{1-\varepsilon}{5\varepsilon}\right)^3$$

Let $N = \left(\frac{1-\varepsilon}{5\varepsilon}\right)^3$, when $n > N$, $n > \left(\frac{1-\varepsilon}{5\varepsilon}\right)^3$

$\Rightarrow \frac{1}{1+5n^{\frac{1}{3}}} < \varepsilon$ for every $\varepsilon > 0$

Since $n \in \mathbb{N}$, $n > 0$, $1+5n^{\frac{1}{3}} > 0 \Rightarrow \frac{1}{1+5n^{\frac{1}{3}}} > 0$

$\Rightarrow 0 < s_n < \varepsilon$ if $n > N$, $N = \left(\frac{1-\varepsilon}{5\varepsilon}\right)^3$

4. For each sequence below, if the sequence converges find its limit or if it diverges explain why it diverges.

(a) $s_n = 3 + 2(-1)^n$

when n is even, $(-1)^n = 1$, when n is odd, $(-1)^n = -1$

if S_n converges, assume $\lim_{n\to\infty} S_n = S$, this for every $\varepsilon > 0$ when have N such that $n > N$ implies $|S_n - S| < \varepsilon$, Take $\varepsilon = \frac{4}{2} > 0$

$|3+2\times 1 - S| < 2$ $\quad |3+2\times(-1) - S| < 2$

$|5 - S| < 2$ $\quad |1 - S| < 2$

$|S-1| < 2$

$2+2 > |5-S| + |S-1| \geq |5-S+S-1| = 4$ (triangular inequality)

$\Rightarrow 4 > 4$ contradiction $\boxed{\text{Hence } S_n \text{ diverges}}$

(b) $s_n = \frac{n^2+5}{10n+3}$

$S_n = \frac{n^2+5}{10n+3} \geq \frac{n^2}{13n} = \frac{n}{13}$ since $10n+3 \leq 13n$, $n^2+5 > n^2$ (since $n \in N$, $n \geq 1$)

for every $M > 0$, Let $N = 13M$, when $n > N$, $n > 13M$

$\frac{n^2+5}{10n+3} \geq \frac{n}{13} > \frac{13M}{13} = M \Rightarrow S_n > M$

Hence, for every $M > 0$, $\exists N$ such that $n > N$ implies $S_n > M$ $\boxed{S_n \text{ diverges}}$

(c) $s_n = n^2/2^n$

The sequence converges, the limit is 0

let $P(n)$ denote $2^n > n^3$

$P(10)$ is true since $2^{10} = 1024 > 10^3 = 1000$

Assume $P(k)$ true, $k \geq 10$

$2^k > k^3$

$2^{k+1} \geq 2k^3 > (k+1)^3$ since $k \geq 10$

by plotting the graph

Hence $\frac{n^2}{2^n} < \frac{n^2}{n^3} = \frac{1}{n}$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, $\frac{n^2}{2^n} > 0$

By Squeeze Theorem, $\lim S_n = 0$.

(d) $s_n = \frac{n^2+5n+6}{4n^2+3n+1}$.

The sequence converges, the limit is $\frac{1}{4}$

$\lim_{n\to\infty} \frac{n^2+5n+6}{4n^2+3n+1} = \lim_{n\to\infty} \frac{1+5\frac{1}{n}+\frac{6}{n^2}}{4+3\frac{1}{n}+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+0+0}{4+0+0} = \frac{1}{4}$