Math 131A, Analysis, Midterm 2

May 17, 8am to May 18, 8am

Name: ____________________________

University ID number: ________________

Write your name and UID number above.
Copy in the box (or on your own papers if you choose to use that to write answers) the following statement and then sign:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signature:______________________ Date:________________

- You **MUST** sign the honor statement. If it is not signed, the test will be given a grade of 0.
- This test is open-book and open-note. It is not timed.
- There are 4 problems, worth a total of 48 points. Show your work legibly with necessary intermediate steps. Points will **NOT** be given to answers without proper justification.
- Submit your answer to Gradescope by the due time. No late submission will be accepted.
- No cheating or collaboration! Getting assistance from others or providing help to others, in any form, is cheating. Cheating of any kind will invalidate the test, and it will be reported to the Office of Dean of Students. The instructor reserves the right to contact you after the exam and ask for additional explanations of your solutions.
- Should you have any questions during the test, contact the instructor immediately.

Problem 1. *12 points*

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two bounded sequences of real numbers. Prove that

$$\liminf a_n + \limsup b_n \leq \limsup(a_n + b_n).$$

Remark. You may use the results in Exercises 9.9, 12.1, 12.4, and 12.5 in Ross without proofs.

Problem 2.

Let $(a_n)_{n\in\mathbb{N}}$ be defined by $a_1 = \sqrt{2}$ and

$$a_{n+1} = \sqrt{2 + a_n} \quad \forall\, n \in \mathbb{N}.$$

(a) [8 points] Prove that $(a_n)_{n\in\mathbb{N}}$ converges;

Hint. Show that it is bounded and monotone.

(b) [4 points] Find $\lim_{n\to+\infty} a_n$. You can use part (a) even if you might not have proved it.

Problem 3.

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers, with $\sum_{n=1}^{\infty} a_n$ being convergent.

(a) [6 points] Show that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ does not converge.

(b) [6 points] Show that there exists a subsequence of $(a_n)_{n\in\mathbb{N}}$, denoted by $(a_{n_k})_{k\in\mathbb{N}}$, such that $\sum_{k=1}^{\infty} a_{n_k}$ converges, and

$$\sum_{k=1}^{\infty} a_{n_k} < \frac{1}{3}\sum_{n=1}^{\infty} a_n.$$

Problem 4.

Let $S \subset \mathbb{R}$, $S \neq \varnothing$. Let f be a real-valued function defined on S. It is known that f is continuous at $x_0 \in S$, and $f(x_0) = 0$.

(a) [6 points] Is it possible that $f(x) \geq 1$ for all $x \in S \backslash \{x_0\}$? If yes, give an example; if not, provide a proof.

(b) [6 points] Suppose g is another real-valued function defined on S. It is known that g is bounded on S, i.e., there exists $M > 0$, such that $|g(x)| \leq M$ for all $x \in S$. Prove that fg is continuous at x_0.

Remark. Just to clarify, here fg denotes the product of f and g instead of their composition. Please also be noted that g is not necessarily continuous.