Math 131A MT

Wilson Jusuf

TOTAL POINTS

64 / 90

QUESTION 1

Problem 1 20 pts

1.1 Part (a) 2 / 2

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect
- - **1 pts** Blank

1.2 Part (b) 1 / 2

- - **0 pts** Correct
- - **2 pts** Incorrect
- ✓ - **1 pts** Blank

1.3 Part (c) 1 / 2

- - **0 pts** Correct
- - **2 pts** Incorrect
- ✓ - **1 pts** Blank

1.4 Part (d) 2 / 2

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect
- - **1 pts** Blank

1.5 Part (e) 2 / 2

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect
- - **1 pts** Blank

1.6 Part (f) 2 / 2

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect
- - **1 pts** Blank

1.7 Part (g) 2 / 2

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect
- - **1 pts** Blank

1.8 Part (h) 0 / 2

- - **0 pts** Correct
- ✓ - **2 pts** Incorrect
- - **1 pts** Blank

1.9 Part (i) 2 / 2

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect
- - **1 pts** Blank

1.10 Part (j) 0 / 2

- - **0 pts** Correct
- ✓ - **2 pts** Incorrect
- - **1 pts** Blank

QUESTION 2

Problem 2 15 pts

2.1 Part (a) 5 / 5

- ✓ - **0 pts** Correct
- - **5 pts** Incorrect
- - **3 pts** Error
- - **2 pts** Error

2.2 Part (b) 5 / 5

- ✓ - **0 pts** Correct
- - **2 pts** Error
- - **3 pts** Error

2.3 Part (c) 4 / 5

- - **0 pts** Correct
- ✓ - **1 pts** Minor Error
- - **2 pts** Error
- - **4 pts** Serious error

- **5 pts** Incorrect
- **3 pts** Error

Need to make argument more rigorous. Specifically: prove that (a_n) is increasing.

QUESTION 3

Problem 3 20 pts

3.1 Part (a) 5 / 5

✓ **- 0 pts Correct**
- **1.5 pts** Only considered natural k
- **3 pts** Misunderstood question
- **5 pts** No solution submitted for Problem 3 Part (a)
- **2 pts** Errors

3.2 Part (b) 5 / 5

✓ **- 0 pts Correct**
- **3 pts** Did not specify construction
- **1 pts** error

3.3 Part (c) 2 / 5

- **0 pts** Correct
- **0 pts** Ok
- **2 pts** Error

✓ **- 3 pts Incorrect**

3.4 Part (d) 2 / 5

- **0 pts** Correct
- **5 pts** Incorrect/Blank
- **3 pts** Misunderstood question
- **2 pts** Incorrect

✓ **- 3 pts Incorrect**

We aren't using decimal expansions in this class.

QUESTION 4

Problem 4 10 pts

4.1 Part (a) 5 / 5

✓ **- 0 pts Correct**
- **3 pts** Quantifier Error
- **1 pts** vague quantifiers
- **2 pts** Unclear quantifiers
- **4 pts** Need to state definition

4.2 Part (b) 2 / 5

- **0 pts** Correct
- **1 pts** Minor error

✓ **- 3 pts Error**
- **4 pts** Serious error
- **5 pts** Incorrect/Blank

This is generally sloppy ("It cannot be negative due to absolute value"). Furthermore, there's no reason for the infimum of a set to be represented in that set (for ex, the set of 1/n for all natural N has infimum 0, but 0 is not 1/n for any natural n), and this means the crucial step in your proof is unjustified.

QUESTION 5

Problem 5 10 pts

5.1 Part (a) 2 / 5

- **0 pts** Correct

✓ **- 2 pts Stated the test assuming the limit of the ratio/root exists**

✓ **- 0.5 pts Did not note that the test implies absolute convergence, not just convergence (waived if only considered positive sequences)**

- **1 pts** In the convergent case, did not say that the ratio/root needs to be less than an \alpha, which is itself less than 1, for all but finitely many terms, and instead said that the ratio/root simply needs to be less than 1 for all but finitely many terms
- **0.5 pts** Did not correctly state for what values of the index n the condition given in the convergent case must hold
- **1 pts** Stated the condition on the general terms of the series in the convergent case without any statement on what values of the index n the condition must hold for
- **0.5 pts** Did not correctly state for what values of

the index n the condition given in the divergent case must hold

- **1 pts** Stated the condition on the general terms of the series in the divergent case without any statement on what values of the index n the condition must hold for
- **0.5 pts** Said that the sequence, rather than the series, converges/diverges.
- **5 pts** Incorrect/Blank

- 0.5 Point adjustment

If the limit is 1, the test is inconclusive, so the inequality in your last statement should not be strict (-0.5pt).

5.2 Part (b) 1 / 5

- **0 pts** Correct
- **2 pts** Error in dealing with (n/(n+1))^n
- ✓ **- 4 pts Only one convergent value is correctly justified**
- **1.5 pts** Substantial errors in the divergent cases
- **3 pts** Both divergent cases are incorrect
- **5 pts** Incorrect / Blank

QUESTION 6

Problem 6 15 pts

6.1 Part (a) 5 / 5

- ✓ **- 0 pts Correct**
- **5 pts** No solution provided for Problem 6 Part (a)
- **2 pts** Incomplete proof
- **3 pts** Invalid argument
- **1 pts** error

6.2 Part (b) 5 / 5

- ✓ **- 0 pts Correct**
- **5 pts** No solution submitted for Problem 6 Part (b)
- **1 pts** Need to prove by induction

6.3 Part (c) 2 / 5

- **0 pts** Correct
- **5 pts** Incorrect with no explanation
- ✓ **- 3 pts Incorrect, the series diverges**
- **5 pts** No solution provided for Problem 6 Part (c)

Math 131a, Summer Session C 2019
Real Analysis
Midterm Exam

August 22, 2019

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient work/explanations, unless the problem explicitly states that no work is necessary. You can not use any notes, books, or electronic devices of any kind during the exam. If you have a question about any particular problem, please raise your hand. At the completion of the exam, please hand the exam booklet. If you have any questions about the grading of the exam, please consult the midterm regrade policy in the course syllabus.

Name: Wilson Jusuf

UID: 404997407

I certify that the work appearing on this exam is completely my own.

Signature: ______________________

Question:	1	2	3	4	5	6	Total
Points:	20	15	20	10	10	15	90
Score:							

1. For each of the following statements, indicate whether they are True or False. **A blank answer will receive 1 point.** [Recall: True means the same thing as "always true" and False means the same thing as "there exists a counterexample".] No work is necessary for this problem.

(a) __F__ The inequality $(27+56)^3 - 27^2 \cdot 28^2 \leq 0$ is true. (2)

(b) ______ Given rational numbers $p < q$, you can always find an *irrational* number $\alpha \in \mathbb{R}$ $\setminus \mathbb{Q}$ such that $p < \alpha < q$. (2)

(c) ______ For each $n \in \mathbb{N}$ define $x_n := \left(1 + \frac{1}{5n}\right)^{5n}$ and $y_n := \left(1 - \frac{1}{7n}\right)^{-7n}$. Furthermore, for each $n \in \mathbb{N}$ define the interval $I_n := [x_n, y_n]$, and consider the set $S := \bigcap_{n=1}^{\infty} I_n$. Then $\inf S < \sup S$. (2)

(d) __F__ Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that

$$\lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} f(-1/n) = 2.$$

Then $\lim_{x\to 0} f(x) = 2$. (2)

(e) __F__ Suppose $(a_n)_{n\geq 1}$ is a strictly decreasing sequence in $\mathbb{R}$ such that $a_n \geq 0$ for every n. Then $\sum(-1)^n a_n$ converges (even though $\sum a_n$ might diverge). (2)

(f) __F__ If $\sum a_n$ converges, then $\sum |a_n|$ converges. (2)

(g) __F__ Suppose $(x_n)_{n\geq 1}$ is a sequence of real numbers without a convergent subsequence. Then there exists a function $f : \mathbb{R} \to [0,1]$ such that the sequence $\left(f(x_n)\right)_{n\geq 1}$ does not have a convergent subsequence. (2)

(h) __T__ Let $I_n := (a_n, b_n)$, where $a_n, b_n \in \mathbb{R}$, $a_n < b_n$ are such that $I_n \supseteq I_{n+1}$ for all n. Then there is some $s \in \mathbb{R}$ such that $s \in I_n$ for all n. (nested intervals lemma) (2)

(i) __F__ Suppose $(a_n)_{n\geq 1}$ is a sequence in $\mathbb{R}$ such that $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$. Then there exists N such that for every $n \geq N$, $a_n \geq a_{n+1}$. (2)

(j) __T__ Suppose $S \subseteq \mathbb{Z}$ is such that $S \neq \emptyset$. Then S has a least element, i.e., there is some $a \in S$ such that for all $b \in S$, $a \leq b$. (well ordering) (2)

2. Do the following:

   (a) State the *Completeness Axiom* (any version). (5)

   (b) State the *Archimedean Property* (any version). (5)

   (c) Suppose $(x_n)_{n\geq 1}$ is a bounded sequence of real numbers. For each $n \geq 1$ define

$$a_n := \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}.$$

   Does the sequence $(a_n)_{n\geq 1}$ converge or diverge? Either prove it always converges or provide a counterexample where it diverges. (5)

(Sup version)
a. For every non-empty set that is bounded above, it has a least upper bound (supremum)

b. for any $a \in \mathbb{R}$, $a>0$, $\exists n \in \mathbb{N}$ such that $n > a$.

c. note that a_n is monotonically increasing because as we advance on n, we have less values to consider a greatest lower bound (infimum).
Since (x_n) is bounded, that means it is bounded from above, and there exists $M \in \mathbb{R}$ such that $x_n \leq M$ $\forall n \in \mathbb{N}$. Hence, $\inf\{x_n, x_{n+1}, \ldots\} \leq M$ and thus a_n is bounded.
By MCT, a_n converges since it is increasing monotone + bounded from above

3. In this problem you will describe how arbitrary real powers b^x are constructed from scratch. You don't have to prove anything, but you do have to give explicit definitions at each stage as to how the relevant number is defined. If you need to use the Completeness Axiom, you should state the specific set you are applying the Completeness Axiom to. At each stage you can (and should) use the concepts defined in previous stages.

   (a) Suppose $b \in \mathbb{R}$ is such that $b > 0$, and $k \in \mathbb{Z}$. How is b^k defined/constructed? (5)

   (b) Suppose $b \in \mathbb{R}$ is such that $b > 0$, and $n \in \mathbb{N}$. How is $b^{1/n}$ defined/constructed? $\sup(E)=y$, $E=\{t: t^n<b\}$ (5)

   (c) Suppose $b \in \mathbb{R}$ is such that $b > 1$, and $q \in \mathbb{Q}$. How is b^q defined/constructed? (5)

   (d) Suppose $b \in \mathbb{R}$ is such that $b > 1$, and $x \in \mathbb{R}$. How is b^x defined/constructed? (5)

   [To be clear on the instructions, for example in part (b) you cannot simply say "$b^{1/n}$ is defined to be the positive number y such that $y^n = b$." You need to say precisely how such a y is found/constructed.]

a). We prove that the function $f(k)=b^k$ obeys the following properties:
$f(k+l)=f(k)f(l)$ for $k,l \in \mathbb{Z}$
$f(k) > f(l)$ iff $k>l$ for $k,l\in\mathbb{Z}$

Thus, we define it as follows: $f(k)=\begin{cases} \underbrace{b\cdots b}_{k \text{ times}} & k>0 \\ 1 & k=0 \\ \underbrace{\frac{1}{b}\cdots\frac{1}{b}}_{|k| \text{ times}} & k<0 \end{cases}$ which obeys the above properties, and is uniquely defined. I.e. if 2 functions f and g agree on these, then $f(k)=g(k)$ $\forall k\in\mathbb{Z}$.

b. Consider the set $E=\{t: t^n<b\}$, $t\in\mathbb{R}$, $n\in\mathbb{N}$. We know that E is non-empty and bounded above, and that t^n is defined for $n\in\mathbb{Z}$ already. By completeness axiom, it must have a supremum. As proven in existence and uniqueness of nth roots, $y=\sup(E)$ is the solution to $y^n=b$, so $y=b^{1/n}$.

c. We prove that there is only one unique exponentiation function $f(q)=b^q$, $q\in\mathbb{Q}$.
Suppose f and g obey for $x,y\in\mathbb{Q}$
$f(x+y)=f(x)f(y)$
$g(x+y)=g(x)g(y)$
$f(x)>f(y)$ iff $x>y$
$g(x)>g(y)$ iff $x>y$
$f(1)=g(1)$
Then we prove $f(x+y)=g(x+y)$
By using fractions x,y that add up to integers, we may use (a) to prove that $f(x)=g(x)$ $\forall x\in\mathbb{Q}$.

d. First, we know that if $x\in\mathbb{Q}$, $x\in\mathbb{R}$ and is defined for part (c).
For $x\in\mathbb{R}\setminus\mathbb{Q}$, we use the decimal expansion of real numbers to approximate some $x\in\mathbb{R}\setminus\mathbb{Q}$ with a sum of fractions. This is defined already by (c).

4. Do the following:

 (a) Suppose $(a_n)_{n\geq 1}$ is a sequence in $\mathbb{R}$ and $a \in \mathbb{R}$. State the definition of "$\lim_{n\to\infty} a_n = a$". (5)

 (b) Do one of the following (circle your choice):

 - Determine (with proof) the limit $\lim_{n\to\infty} n^{1/n}$.
 - Determine (with proof) the limit $\lim_{n\to\infty} a^n$, where $|a| < 1$.

 Since these were done in class, you can't just say as an answer "we did this in class", you actually have to give the derivation. You are allowed to use things such as: the definition of convergence, the squeeze lemma for sequences, various important inequalities, other more basic limits which are established before these ones, etc. (5)

(where m is the starting index of the sequence)

a. $\lim_{n\to\infty} a_n = a$ means that for every $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$, there exists $N \geq m$ such that $|a_n - a| < \varepsilon$ holds for all $n \geq N$.

~~We want to show that for every $\varepsilon > 0$, $\exists N \geq 1$ s.t. for $n \geq N$, $|a^n - 0| < \varepsilon$~~

~~b. (i.e. a^n converges to 0).~~

~~we know that $|a^n - 0| < 1$ Since $|a| < 1 \Rightarrow |a|^n < 1$, (note that $|a^n| = |a|^n$ in this case)~~

~~So for $\varepsilon \geq 1$, we know that $|a^n - 0| < \varepsilon$ holds.~~

~~For $0 < \varepsilon < 1$, Note that for any $\varepsilon > 0$, $|a|^n = \varepsilon$ has a solution $|a|$ by the existence and uniqueness of nth roots.~~

~~Since $|a| < 1$, $|a|^{n+1} < |a|^n = \varepsilon$ (as per above).~~

~~Which means for $N = n+1$, we are guaranteed that $|a^k - 0| < \varepsilon$ for $k \geq N$.~~

We know that $|a|^n$ is decreasing (since $|a| < 1 \Rightarrow |a|^{n+1} < |a|^n$), and bounded at 0 (Since $0 \leq |a| < 1$). Thus it converges by MCT, and thus 0 is the limit.

We must prove that the infimum of set of $|a|^n$ values must be 0. It can't be negative due to absolute value.

Suppose towards contradiction that it is $\neq 0$.

It can't be ≥ 1 Since $0 < |a| < 1 \Rightarrow 0 < |a|^n < 1$ by power ineq.

Suppose that $\min\{|a|^n\} = \varepsilon$, where $0 < \varepsilon < 1$, and $\varepsilon \in \{|a|^n : n \in \mathbb{N}\}$. notice how $\varepsilon |a| < \varepsilon$, and

$\varepsilon |a| \in \{|a|^n : n \in \mathbb{N}\}$, which is a contradiction.

Hence $\lim_{n\to\infty} a^n = 0$

5. (a) Do one of the following (circle your choice):
   - State the *Root Test*
   - State the *Ratio Test*

   For full credit you must state the version of the test which was proved in class. If you state a weaker version then you will lose points. (5)

   (b) For the following series, provide (with justification) two values of x for which the series converges and two values of x for which the series diverges: (5)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n.$$

Cauchy Condensation

$\sum_n 2^n a_{2^n} \Leftarrow\Rightarrow \sum_m a_m$ converges

root test

a. Suppose $0 \le \alpha < 1$. ~~If the~~ ~~let~~ Consider some series $S_n = \sum a_n$.

if $|a_n|^{1/n} \to \alpha$, then S_n converges.

if $\alpha \ge 1$, then S_n diverges.

b. when $x = 0$, the series converges to 0.

$\sum_{n=1}^{\infty} \frac{n!}{n^n} 0^n = \sum 0 = 0.$

when $x = 1$,

$\sum_{n=1}^{\infty} \frac{n!}{n^n} 1^n = \sum \frac{n!}{n^n} = \sum \frac{(n)(n-1)(n-2)\cdots 1}{n\, n\, n \cdots n} > \sum_n \frac{1}{n}$

note that $\sum_n \frac{1}{n^n} < \sum_n \frac{1}{n}$ yet $\sum \frac{1}{n}$ diverges

6. Suppose $L : (0, +\infty) \to \mathbb{R}$ is a function which enjoys the following properties:

   (i) for every $a, b \in (0, +\infty)$, $L(ab) = L(a) + L(b)$;

   (ii) for every $a, b \in (0, +\infty)$, if $a < b$, then $L(a) < L(b)$.

   Prove the following things:

   (a) Show that $L(n) > 0$ for every $n \in \mathbb{N}$ such that $n \geq 2$. (5)

   (b) Suppose $b > 1$. Show that $L(b^n) = nL(b)$ for every $n \in \mathbb{N}$. (5)

   (c) Determine (with valid proof using only techniques taught in class) whether or not the following series converges or diverges: (5)

$$\sum_{n=2}^{\infty} \frac{1}{nL(n)}.$$

a.) prove by induction
base case

$L(2) = L(1) + L(2)$

for this to hold, it must be that $L(1) = 0$

by II, $1 < 2$, so $L(1) = 0 < L(2)$.

inductive step: assume $L(n) > 0$. for $L(n+1)$, note that $n+1 > n$, so by (II), $L(n+1) > L(n) > 0$.

b.) $L(b^n) = L(b(b^{n-1})) = L(b) + L(b^{n-1}) = L(b) + L(b) + L(b^{n-2}) \dots$ etc.

prove this inductively.

Base case $n=1$

$L(b) = 1L(b) = L(b)$

inductive step. assume that $L(b^n) = nL(b)$. for $n+1$, $L(b^{n+1}) = L(b) + L(b^n)$ (by (i))

$= L(b) + nL(b)$ (by assumption)

$= (n+1)L(b)$ □

c.) consider $\sum_{n=2}^{N} \frac{1}{nL(n)} = \frac{1}{2L(2)} + \frac{1}{3L(3)} + \frac{1}{4L(4)} + \frac{1}{5L(5)} + \dots$

$= \sum_{n=1}^{N} \left(\frac{1}{2^n nL(2)} + \frac{1}{3^n nL(3)} + \frac{1}{5^n nL(5)} + \frac{1}{6^n nL(6)} + \dots \right) = \sum_{n=1}^{N} \left(\sum_{K \text{ is not a square number}}^{\infty} \frac{1}{k^n nL(k)} \right)$

$< \sum_{n=1}^{N} \left(\frac{1}{2^n nL(2)} + \frac{1}{2^n nL(3)} + \dots \right)$

$= \sum_{n=1}^{N} \frac{1}{2^n n} \left(\frac{1}{L(2)} + \frac{1}{L(3)} + \dots \right)$

$< \sum_{n=1}^{N} \frac{1}{2^n n} \left(\frac{N-1}{L(2)} \right)$ which converges. Hence, it converges by comparison

$\frac{1}{2L(2)} + \frac{1}{3L(3)} + \frac{1}{4L(4)} + \frac{1}{5L(5)}$

$= \frac{1}{2L(2)} + \frac{1}{3L(3)} + \frac{1}{8L(2)} + \frac{1}{5L(5)} + \frac{1}{6L(6)} + \frac{1}{7L(7)} + \frac{1}{8L(8)} + \frac{1}{27L(3)}$

$\frac{1}{24L(2)}$

pow 2: $\frac{1}{2^n nL(2)}$