But $\sum \frac{1}{2n}$ is not convergent by integral test. $\Longrightarrow \sum \frac{1}{n^{1+\frac{1}{n}}}$ $\left(\geq \sum \frac{1}{2n} \right)$ is not conv't.

MATH 131A Midterm II, Fall 2019

Name:

Justify All Your Answers.

Problem 1. (5)

(i) Decide for which p, the series $\Sigma_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent.
(ii) Decide if the series $\Sigma_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ is convergent.

(i) $$\int_2^\infty f(x) = \int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_2^\infty \frac{1}{(\ln x)^p}\,d\ln x$$

2.5 pts

$$= \begin{cases} (\ln x)^{-p+1}/_{1-p} \Big|_2^\infty & p \neq 1 \\ \ln(\ln x) \Big|_2^\infty & p = 1. \end{cases}$$

If $p \leq 1$. $(\ln x)^{1-p} \Big|_2^\infty = \infty$
$\ln(\ln x) \Big|_2^\infty = \infty$

$\Longrightarrow$ not convergent.

If $p > 1$. $(\ln x)^{-p+1}/_{1-p} = \frac{1}{(\ln 2)^{p-1}}$

$\Longrightarrow$ convergent.

(ii) It is not convergent: since $\lim n^{\frac{1}{n}} = 1$.

2.5 pts

$\therefore$ for n large enough, $n^{\frac{1}{n}} \leq 2$ &

$$\frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n \cdot n^{\frac{1}{n}}} \geq \frac{1}{2n}$$

Let $x_n = n$ & $y_n = n + \frac{1}{n}$. Then $|x_n - y_n| = \frac{1}{n} \xrightarrow{n\to\infty} 0$

(*) But $|f(x_n) - f(y_n)| = |n^3 - (n+\frac{1}{n})^3| = |n^3 - (n^3 + 3n + 3/n + \frac{1}{n^3})|$

$= |3n + 3/n + \frac{1}{n^3}| \xrightarrow{n\to\infty} \infty$.

Thus for $\varepsilon = 1$ & any $\delta > 0$, for n sufficiently large, even $|x_n - y_n| = \frac{1}{n} < \delta$, $|f(x_n) - f(y_n)| > 1 (= \varepsilon)$ for x_n & y_n above $\square$

Problem 2. (5)

(i) Let $y = f(x)$ be a function on $\mathbf{R}^1$ defined by $f(x) = \cos\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Using the definition only, show that f is not continuous at $x = 0$.

(ii) Let $y = f(x)$ be a function on $\mathbf{R}^1$ defined by $f(x) = x\cos\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Using the definition only, show that f is continuous at $x = 0$.

(iii) Is the function $f(x) = x^3$ is uniformly continuous on a finite interval $[a, b]$? Is it uniformly continuous on $\mathbf{R}^1$?

(i) Consider $x_n = \frac{1}{2\pi n}$. Then $(x_n) \xrightarrow{n\to\infty} 0$

But $f(x_n) = \cos(2\pi n) = 1$

2 pts

& $\lim f(x_n) = 1 \neq f(0) = 0$.

(ii) For $\varepsilon > 0$,

since $|f(x) - f(0)|$

$= |x \cos\frac{1}{x}| \leq |x| |\cos(\frac{1}{x})| \leq |x|$,

1 pt

Let $\delta = \varepsilon$. Then for $|x - 0| = |x| < \delta$,

$|f(x) - f(0)| \leq |x| < \varepsilon$.

2

(iii) Since $f(x) = x^3$ is cont on $\mathbb{R}^1$, it is uniformly cont on any finite interval $[a, b]$.

2 pts

But f is not uniformly cont on $\mathbb{R}^1$: (*)

Problem 3. (5)

(i) Let $f(x) = x^3 + x^2 - 100x + 2$. Show that the equation $f(x) = 0$ has at least one real root.

(ii) How many real roots of the equation $f(x) = 0$ in (i) has?

(i) Since $\lim_{x \to \pm\infty} f(x) = \pm\infty$.

$\therefore \exists M > 0$ s.t if $x > M$

$f(x) > 0$ & $x < -M$

2.5 pts $f(x) < 0$. Since f is cont.

$\therefore \exists x_0 \in (-M, M)$ s.t

$f(x_0) = 0$ (intermediate value thm).

(ii) $f(0) = 2 > 0$, $f(1) = -96 < 0$

Assume that $M > 1$ in (i)

Then $\exists x_1 < -M < 0$ s.t $f(x_1) < 0$

2.5 pts & $x_2 > M > 1$ s.t $f(x_2) > 0$

$\therefore \exists$ $y_0 \in (x_1, 0)$ s.t $f(y_0) = 0$ $\Rightarrow f$ has

$y_1 \in (0, 1)$ s.t $f(y_1) = 0$ & 3 real

$y_2 \in (1, x_2)$ s.t $f(y_2) = 0$ roots.

$(*) \Longrightarrow \{ f(x_n) \mid n > N \}$ is bdd above

Contradicts to $\lim f(x_n) = \infty$. $\square$

Problem 4. (5)

(a) Let $f : [a, b] \to \mathbf{R}^1$ be a continuous function show that the image $f([a, b])$ is either a closed interval or a single point.

(b) Let $f : [a, b] \to \mathbf{R}$ be a function that is upper semi-continuous at all points of $[a, b]$, prove or disprove that f is bounded above. Here f with domain $S \subset \mathbf{R}$ is said to be upper semi-continuous at $x_0 \in S$ if for any given $\epsilon > 0$ there exists a $\delta > 0$, such that when $x \in S$ with $|x - x_0| < \delta$, $f(x) < f(x_0) + \epsilon$.

(a) Assume that f is not a constant

Then by intermediate value thm

$f([a,b])$ is a interval. Since

f assumes max. & min values, say

2 pts

at pts x_0 & $x_1 \in [a,b]$ respectively.

$\Longrightarrow \quad f([a,b]) = [f(x_1), f(x_0)]$

is a closed interval.

(b) The statement is true:

If f is not bdd above $\Longrightarrow$ $\exists\, x_n \in [a,b]$

s.t $f(x_n) \xrightarrow{n \to \infty} \infty$. After taking a subsequence

3 pts

WLOG $\lim x_n = y_0 \in [a,b]$ (B-W thm)

For $\varepsilon = 1$, $\exists\, \delta > 0$ s.t when $|x - y_0| < \delta$, $f(x) < f(y_0) + \varepsilon$

by upper semi-cont. For such a δ, $\exists N$ s.t $(*)$

when $n > N$, $|x_n - y_0| < \delta$, hence $f(x_n) < f(y_0) + \varepsilon$.