MATH 131A Midterm I, Fall 2019

Name:

Justify All Your Answers.

Problem 1. (5)

(i) Decide if $\sqrt{5-\sqrt{2}}$ is a rational number.

(ii) Prove your conclusion in (i).

1 pt (i) $r = \sqrt{5-\sqrt{2}}$ is not rational

(ii) $r^2 - 5 = -\sqrt{2}$ $(*)$

If r is rat'l $\Rightarrow$ $r^2 - 5$ is rat'l

$\Rightarrow$ $-\sqrt{2}$ is rat'l.

4 pts.

Method 2: $(*) \Rightarrow r^4 - 10r^2 - 23 = 0$

If $r = \frac{p}{q}$ $\Rightarrow$ $q \mid 1$ & $p \mid 23$

$\Rightarrow$ $r = \pm 1, \pm 23$ But for

such r, $r^4 - 10r^2 - 23 \neq 0$.

Problem 2. (5)

(i)Using the theorem on limits find $lim s_n$. Here $s_n = \frac{n^2+2n+6}{n^2-5}$.

(ii) Using the definition only give a formal proof that the sequence $\{s_n\}_{n=1}^{\infty}$ in (i) above is convergent to the limit you have found.

(iii) Using the definition only show that $s_n = 1 + (-1)^n 2$ is not convergent.

(i) $\lim s_n = \lim \frac{(1+\frac{2}{n}+\frac{6}{n^2})\cdot n^2}{(1-\frac{5}{n^2})\cdot n^2}$

1 pt $= \frac{\lim(1+\frac{2}{n}+\frac{6}{n^2})}{\lim(1-\frac{5}{n^2})} = \boxed{1}$

(ii) step I: Find $N = N(\varepsilon)$, for $n > N$

$|s_n - 1| = \left|\frac{n^2+2n+6}{n^2-5} - 1\right|$

2 pts $= \frac{|2n+11|}{|n^2-5|} \le \frac{3n}{\frac{1}{2}n^2}$ if (1) $n \ge 11$ (2) $\frac{1}{2}n^2 \ge 5 \iff n \ge \sqrt{10}$

$= \frac{6}{n} < \frac{6}{N} < \varepsilon$ if $N > \left[\frac{6}{\varepsilon}\right]$ = integral part of $\frac{6}{\varepsilon}$ + 1.

step II: Set $N = \max\left\{\left[\frac{6}{\varepsilon}\right], \cancel{11}\right\}$

Then for given $\varepsilon > 0$, by step I, for $n > N$,

$|s_n - 1| < \varepsilon$.

2 pts (iii) $s_{2k} = 1+2 = 3$ & $s_{2k+1} = 1-2 = -1$. If (s_n) converges to s $\Rightarrow$ $\forall \varepsilon > 0$, $\exists N$ s.t. $2k$ & $2k+1 > N$, $4 = |s_{2k} - s_{2k+1}|$ $\le |s_{2k} - s| + |s_{2k+1} - s| < 2\varepsilon$. A contradiction if $\varepsilon < 2$.

Then $\limsup s_n = \limsup t_n = 1 \Rightarrow \limsup s_n \cdot \limsup t_n = 1$.

But $s_n \cdot t_n = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \ldots \frac{1}{2n}, \frac{1}{2n}\}$

$\therefore \lim s_n t_n = \limsup s_n t_n = 0$

$\square$.

Problem 3. (5)

(i) Let S be a subset of $\mathbf{R}$ bounded above but without maximum. Using the definition only show that the set T defined to be $T =: \{x| -x \in S\}$ is a subset of $\mathbf{R}$ bounded below but without minimum.

(ii) Prove or disprove that if $(s)_{n\in\mathbf{N}}$ and $(t)_{n\in\mathbf{N}}$ are two bounded sequences with $s_n > 0$ and $t_n > 0$, then $limsup(s_n t_n) = limsups_n \cdot limsupt_n$.

(i) Let M be an upper bd of S.

Then $-M$ is an lower bd of T:

$\forall x \in T, \quad -x \in S \Rightarrow -x \leq M$.

2 pts $\Leftrightarrow x \geq -M$.

Assume that $-x_0 \in T$ is the minimal of T.

Then $x_0 = -(-x_0) \in S$ is the maximum

of S: for $\forall x \in S$, $-x \in T$ w/ $-x \geq -x_0$

$\Leftrightarrow x \leq x_0$

$\square$.

(ii) The statement is not true:

Let $(s_n) = (1, \frac{1}{2}, 1, \frac{1}{4}, \ldots 1, \frac{1}{2n} \ldots)$

3 pts w/ $s_{2k+1} = 1 \quad s_{2k} = \frac{1}{2k}$

& $(t_n) = (\frac{1}{2}, 1, \frac{1}{4}, 1 \ldots \frac{1}{2n}, 1 \ldots)$

w/ $t_{2k+1} = \frac{1}{2k+2}, \quad t_{2k} = 1$.

Problem 4. (5)

The sequence $\{s_n\}_{n=1}^{n}$ is defined inductively by the condition $s_{n+1} = \frac{1}{6}(s_n + 3)$ with $s_1 = 1$.

(i)Show that $\{s_n\}_{n=1}^{n}$ is decreasing and bounded below without using the explicit expression of s_n.
(ii) Find $lims_n$.

(i) $s_2 = \frac{1}{6}(s_1+3) = \frac{1}{6}(1+3) = \frac{2}{3}$

$\therefore \quad s_2 < s_1 \quad \& \quad s_2 - s_1 = -\frac{1}{3} < 0$

Assume that $s_n < s_{n-1}$ so that

$s_n - s_{n-1} < 0$ Then

3 pts

$s_{n+1} - s_n = \frac{1}{6}(s_n - s_{n-1}) < 0$

$\Longrightarrow \quad s_{n+1} < s_n$

By induct $\boxed{(s_n) \text{ is decreasing.}}$

- $s_n > 0$: $s_1 = 1 > 0$.

Assume that $s_n > 0$. Then

$s_{n+1} = \frac{1}{6}(s_n+3) > \frac{1}{6} \cdot 3 = \frac{1}{2} > 0$.

$\Longrightarrow$ (s_n) is bdded below.

(ii) By (i) $\lim s_n = s$ exists

2 pts

$\therefore \quad s = \lim s_{n+1} = \lim \frac{1}{6}(s_n+3) = \frac{1}{6}(s+3)$

$\Longrightarrow \quad 5s = 3.$ $\boxed{s = 3/5}$