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MATH 131A Midterm I, Fall 2019

Name: Neil Prajapati

Justify All Your Answers.

Problem 1. (5)

(i) Decide if $\sqrt{5-\sqrt{2}}$ is a rational number.

(ii) Prove your conclusion in (i).

i) It is not rational.

ii) Let $x = \sqrt{5-\sqrt{2}}$

$x^2 = 5 - \sqrt{2}$

$x^2 - 5 = -\sqrt{2}$

$x^4 - 10x^2 + 25 = 2$

$x^4 - 10x^2 + 23 = 0$

$\frac{p}{q} = \frac{\pm 1}{\pm 1}, \pm 23$

$(\pm 23)^4 - 10(\pm 23)^2 + 23 \neq 0$

$1^4 - 10(1) + 23 = 14 \neq 0$

Thus x cannot be rational by the rational roots theorem.

Problem 2. (5)

(i)Using the theorem on limits find $lim s_n$. Here $s_n = \frac{n^2+2n+6}{n^2-5}$.

(ii) Using the definition only give a formal proof that the sequence $\{s_n\}_{n=1}^{\infty}$ in (i) above is convergent to the limit you have found.

(iii) Using the definition only show that $s_n = 1 + (-1)^n 2$ is not convergent.

i) $\lim s_n = \lim \frac{n^2+2n+6}{n^2-5} = \lim \frac{1 + 2/n + 6/n^2}{1 - 5/n^2} = \lim$

$= \frac{\lim 1 + \lim 2/n + \lim 6/n^2}{\lim 1 + \lim -5/n^2} = \frac{1}{1} = 1$ ✓

ii) $\left|\frac{n^2+2n+6}{n^2-5} - 1\right| = \left|\frac{n^2+2n+6-n^2+5}{n^2-5}\right| = \left|\frac{2n+11}{n^2-5}\right|$

$\left|\frac{2n+11}{n^2-5}\right| \leq \left|\frac{13n}{\frac{1}{2}n^2}\right|$ when $n \geq 4$

$\left|\frac{2n+11}{n^2-5}\right| \leq \left|\frac{26}{n}\right|$ ✓

let $\varepsilon > 0$ and let $N = \max\{\frac{26}{\varepsilon}, 4\}$. If $n > N$, then $n > \frac{26}{\varepsilon}$ so $\varepsilon > \frac{26}{n}$. Since $\frac{26}{n} \geq \left|\frac{n^2+2n+6}{n^2-5} - 1\right|$ then $\left|\frac{n^2+2n+6}{n^2+5} - 1\right| < \varepsilon$. Thus $\boxed{\lim s_n = 1}$

iii) All even terms can be written as $s_{2k} = 3$, $k \in \mathbb{N}$. Similarly, $s_{2k+1} = -1$ for $k \in \mathbb{N}$. Assume $\lim s_n = s$ then $|s_n - s| < \varepsilon/2$ $\varepsilon > 0$ and $n > N$. So

$|s_{2k+1} - s| < \frac{\varepsilon}{2}$ $\quad |s_{2k} - s| < \frac{\varepsilon}{2}$ $\quad 2k > N$

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$|s_{2k+1} - s| + |s_{2k} - s| < \varepsilon$ $\quad -1$

$|s_{2k+1} - s| + |s - s_{2k}| < \varepsilon$ $\quad$ true if put 2ε

$|s_{2k+1} - s_{2k}| < \varepsilon$

$4 < \varepsilon$ $\quad \forall \varepsilon > 0$

which is false. Thus $\lim s_n$ does not exist

Problem 3. (5)

(i) Let S be a subset of $\mathbf{R}$ bounded above but without maximum. Using the definition only show that the set T defined to be $T =: \{x| -x \in S\}$ is a subset of $\mathbf{R}$ bounded below but without minimum.

(ii) Prove or disprove that if $(s)_{n\in\mathbf{N}}$ and $(t)_{n\in\mathbf{N}}$ are two bounded sequences with $s_n > 0$ and $t_n > 0$, then $limsup(s_n t_n) = limsup s_n \cdot limsup t_n$.

i) By the Completeness Axiom, If S is bounded above, it has a least upper bound $S_0 = \sup S$. Since S_0 is an upperbound $x \leq S_0$ for $\forall x \in S$.

$x \leq S_0$

$-x \geq -S_0$ for $\forall x \in S$

but $\{-x \mid x \in S\}$ is the definition of T. So $\forall t \in T$, $t > -S_0$. Thus T is bounded below.

$S_0 \notin S$ by the assumption. Let's assume $t_0 = -S_0 \in T$.

If $t_0 \in T$, then, $t_0 = -s$ where $s \in S$. $s = -t_0 = --S_0 = S_0$

So $S_0 \in S$. Contradiction. Thus, $t_0 \notin T$.

So T is bounded below and does not have a minimum.

✓

ii) Let's assume $\sup\{s_n t_n : n > N\} > \sup\{s_n : n > N\} \cdot \sup\{t_n : n > N\}$

if $>$, then we could plug in the same n for both $\sup\{s_n : n > N\}$ and $\sup\{t_n : n > N\}$. Thus, our statement is False and $\sup\{s_n t_n : n > N\} \leq \sup\{s_n : n > N\} \cdot \sup\{t_n : n > N\}$.

X -3

can be <

Problem 4. (5)

The sequence $\{s_n\}_{n=1}^{n}$ is defined inductively by the condition $s_{n+1} = \frac{1}{6}(s_n + 3)$ with $s_1 = 1$.

(i)Show that $\{s_n\}_{n=1}^{n}$ is decreasing and bounded below without using the explicit expression of s_n.
(ii) Find $\lim s_n$.

i) Decreasing:

Base case.

$S_1 \geq S_2$

$1 \geq \frac{5}{6}$ ✓

Inductive case: Assume $S_{k-1} \geq S_k$

$S_{k-1} \geq S_k$

$S_{k-1} + 3 \geq S_k + 3$

$\frac{1}{6}(S_{k-1} + 3) \geq \frac{1}{6}(S_k + 3)$ ✓

$S_k \geq S_{k+1}$

Thus by math induction, the sequence is decreasing.

bounded. Since the sequence is decreasing, we know

$S_n \geq S_{n+1}$

$S_n \geq \frac{1}{6}(S_n + 3)$ this not valid

-1

$6S_n \geq S_n + 3$

$5S_n \geq 3$

$S_n \geq 3/5$ for $\forall n \in \mathbb{N}$.

Thus $(S_n)_{n \in \mathbb{N}}$ is bounded below.

ii) $S = \lim \frac{1}{6}(S_n + 3)$

at very large n, $S_n \to S$. Thus

✓ $S = \frac{1}{6}(S + 3)$

$6S = S + 3$

$5S = 3$

$\boxed{S = 3/5}$