22W-MATH-131A-LEC-4 Final

[redacted]

TOTAL POINTS

53 / 60

QUESTION 1

Problem 1 10 pts

1.1 **2 / 5**

+ **1 pts** Valid strategy (e.g., showing that $$f$$ has a continuous extension $$\widetilde{f}:[0,1]\to \mathbb{R}$$)
+ **1 pts** Correct extension for $$\widetilde{f}$$ (defined $$\widetilde{f}(0)=0$$)
+ **2 pts** Correct proof that $$\widetilde{f}$$ is continuous at $$0$$ (need to use squeeze theorem or delta-epsilon definition)
+ **1 pts** Coherent proof
+ **1 pts** Partially correct proof that $$\widetilde{f}$$ is continuous at $$0$$, or gave some of the argument but left out important details

+ **2 Point adjustment**

Attempted a proof from definition but made significant errors

1. For $$\epsilon$$ small this will be negative
2. You'll want to factor out $$x-y$$ from all parts, so that it is $$|x-y|$$ times something else, rather than this

1.2 **4 / 5**

+ **5 pts** Correct

✓ + **4 pts Minor gap or error**

+ **3 pts** Significant gap or error
+ **1 pts** Attempted problem but minimal progress

3. You should describe how to pick this $$x$$ explicitly

QUESTION 2

2 Problem 2 7 / 10

Part 1

✓ + **1 pts Correct strategy: aimed to show that $$\lim_{x\to 0} f(x) = f(0)$$**

+ **2 pts** Fully correct proof (e.g., squeeze theorem)

✓ + **1 pts Partialy correct proof**

Part 2

✓ + **1 pts Correct strategy: aimed to show that $$\lim_{x\to 0} \frac{g(x)-g(1)}{x-1}$$ existed**

+ **2 pts** Fully correct proof
+ **1 pts** Partially correct proof

✓ + **1 pts Claimed that $$\lim_{x\in \mathbb{Q}} = \lim_{x\in\mathbb{I}}$$ was sufficient for existence of ordinary limit (this is true but not obvious)**

Part 3

✓ + **2 pts Correct strategy: demonstrated there is sequence(s) of points in the domain with $$x\to 0$$ yet tending towards different output values**

+ **2 pts** Fully correct proof

✓ + **1 pts Partially correct proof**

4. This inequality is not true for $$x$$ negative
5. Not rigorous

QUESTION 3

3 Problem 3 10 / 10

✓ **- 0 pts Correct**

- **7 pts** Solved problem for specific example rather than in the general case
- **6 pts** Non-rigorous use of infinities (e.g., took a limit in the MVT expression $$f(x)-f(y) = f'(c)(x-y)$$ or something similar)
- **8 pts** Invalid proof strategy

QUESTION 4

4 Problem 4 **10 / 10**

Parts 1, 2

- **- 1 pts** Incorrect $$M(f, [\frac{k-1}{n}, \frac{k}{n}])$$
- **- 1 pts** Incorrect $$m(f, [\frac{k-1}{n}, \frac{k}{n}])$$
- **- 2 pts** Incorrect $$U(f, P_n)$$ with incorrect definition
- **- 1 pts** Incorrect $$U(f,P_n)$$ but correct definition
- **- 2 pts** Incorrect $$L(f, P_n)$$ with incorrect definition
- **- 1 pts** Incorrect $$L(f, P_n)$$ but correct definition

Part 3

- **- 1 pts** Incorrect $$\lim_{n\to\infty}U(f, P_n)$$
- **- 1 pts** Incorrect $$\lim_{n\to\infty}L(f, P_n)$$

Part 4

- **- 1 pts** Not showing $$\lim_{n\to\infty}(U(f,P_n)-L(f,P_n))=0$$
- **- 1 pts** Insufficient / unclear argument to conclude

✓ **- 0 pts** Correct

QUESTION 5

5 Problem 5 **10 / 10**

Part 1

- **- 1 pts** Not correctly apply the MVT
- **- 1 pts** Incorrect argument to conclude

Part 2

- **- 1 pts** Not correctly apply the EVT to find $$x_k,y_k\in [\frac{k-1}{n},\frac{k}{n}]$$ such that $$f(x_k)=M, f(y_k)=m$$, or something similar
- **- 1 pts** Not correctly apply Part 1
- **- 1 pts** Incorrect argument to conclude

Part 3

- **- 1 pts** Incorrect formula for $$U,L$$
- **- 2 pts** Incorrect argument to conclude
- **- 1 pts** Minor mistake

Part 4

- **- 1 pts** Not show $$\lim_{n\to\infty}(U(f, P_n)-L(f,P_n))=0$$ or something similar
- **- 1 pts** Incorrect argument to conclude

✓ **- 0 pts** Correct

QUESTION 6

6 Problem 6 **10 / 10**

Part 1

- **- 1 pts** $$\lim_{h\to 0}\frac{f(h^2)-f(0)}{h^2}$$ does not exist
- **- 1 pts** $$f$$ is differentiable at $$0$$
- **- 1 pts** Incorrect/insufficient/unclear proof

Part 2

- **- 1 pts** $$g$$ is integrable / does not have the domain $$[0,1]$$
- **- 1 pts** $$g^2$$ is not integrable
- **- 1 pts** Incorrect/insufficient/unclear proof

Part 3

- **- 1 pts** $$h$$ is continuous at some $$x\in \R$$
- **- 1 pts** $$h^2$$ is not differentiable at some $$x\in \R$$
- **- 1 pts** Incorrect/insufficient/unclear proof for discontinuity of $$h$$
- **- 1 pts** Incorrect/insufficient/unclear proof for differentiability of $$h^2$$

✓ **- 0 pts** Correct

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I certify on my honor that I have neither give nor received any help, or used any non-permitted resources, while completing this evaluation. -Trevor Guo

1 Question 1

1. Define a function $f : (0, 1) \to \mathbb{R}$ by $f(x) = x^3 \cos(1/x) + x^2 \sin(1/x) + x$. Show that f is uniformly continuous.

   We want to show that $\forall \epsilon > 0, \exists \delta > 0$ such that for any $x, y \in (0, 1)$

   $$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

   $$|f(x) - f(y)|$$
   $$= |x^3 \cos(1/x) + x^2 \sin(1/x) + x - y^3 \cos(1/y) - y^2 \sin(1/y) - y|$$

   By triangle inequality

   $$\leq |x^3 \cos(1/x) - y^3 \cos(1/y)| + |x^2 \sin(1/x) - y^2 \sin(1/y)| + |x - y|$$

   cos and sin are bounded

   $$-1 \leq \cos x, \sin x \leq 1$$

   We can select $x, y \in \mathbb{R}$ such that

   $$\cos(1/x) = 1, \cos(1/y) = -1$$

   Then
   $$|x^3 \cos(1/x) - y^3 \cos(1/y)| = |x^3 - (-y^3)| = |x^3 + y^3|$$
   The same could be done to set the equation equal to $|-x^3 - y^3| = |x^3 + y^3|$
   So
   $$|x^3 \cos(1/x) - y^3 \cos(1/y)| \leq |x^3 + y^3|$$
   For the same reason,

   $$|x^2 \sin(1/x) - y^2 \sin(1/y) \leq |x^2 + y^2|$$

   Thus
   $$|f(x) - f(y)| \leq |x^3 + y^3| + |x^2 + y^2| + |x - y|$$
   Because the domain is $(0, 1)$

   $$|x^3 + y^3| < 2, |x^2 + y^2| < 2$$

   So
   $$|f(x) - f(y)| < 2 + 2 + \delta = 4 + \delta$$

   Proof. Let $\delta = \epsilon - 4$. Then for $x, y \in (0, 1)$ if $|x - y| < \delta$, we have that $|f(x) - f(y)| < 4 + \delta = \epsilon$. Thus we have proven f is uniformly continuous on $(0, 1)$. POG

2. Define a function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = -x^3 - x^2 + 1$. Show that g is not uniformly continuous.

   Proof. To show that g is not uniformly continuous, we want to show it violates the conditions for uniform continuity. Specifically, there exists an $\epsilon > 0$ such that for each $\delta > 0$, $|x - y| < \delta$ implies $|f(x) - f(y)| > \epsilon$. Fix $\epsilon = 1$, and let $y = x + \frac{\delta}{2}$. We always have that

   $$|x - y| = |x - x - \delta/2| = \delta/2 < \delta$$

   Suppose for contradiction that g is uniformly continuous.

   $$|f(x) - f(y)| = |-x^3 - x^2 + 1 + y^3 + y^2 - 1|$$
   $$= \left|3x^2\frac{\delta}{2} + 3x(\frac{\delta}{2})^2 + (\frac{\delta}{2})^3 + x\delta + (\frac{\delta}{2})^2\right| < 1$$

   However, this is a contradiction because we can choose a large enough x such that the expression is greater than 1. Thus g is not uniformly continuous. POG

1.1 2 / 5

+ **1 pts** Valid strategy (e.g., showing that $$f$$ has a continuous extension $$\widetilde{f}:[0,1]\to \mathbb{R}$$)
+ **1 pts** Correct extension for $$\widetilde{f}$$ (defined $$\widetilde{f}(0)=0$$)
+ **2 pts** Correct proof that $$\widetilde{f}$$ is continuous at $$0$$ (need to use squeeze theorem or delta-epsilon definition)
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Attempted a proof from definition but made significant errors

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$$|x - y| = |x - x - \delta/2| = \delta/2 < \delta$$

   Suppose for contradiction that g is uniformly continuous.

$$|f(x) - f(y)| = | - x^3 - x^2 + 1 + y^3 + y^2 - 1|$$

$$= \left|3x^2\frac{\delta}{2} + 3x(\frac{\delta}{2})^2 + (\frac{\delta}{2})^3 + x\delta + (\frac{\delta}{2})^2\right| < 1$$

   However, this is a contradiction because we can choose a large enough x such that the expression is greater than 1. Thus g is not uniformly continuous. POG

3

1.2 4 / 5

- \+ **5 pts** Correct
- ✓ + **4 pts** **Minor gap or error**
- \+ **3 pts** Significant gap or error
- \+ **1 pts** Attempted problem but minimal progress

③ You should describe how to pick this $$x$$ explicitly

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2 Question 2

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x^2 - x & x \notin \mathbb{Q} \end{cases}; g(x) = \begin{cases} x^2 + x & x \in \mathbb{Q} \\ 3x - 1 & x \notin \mathbb{Q} \end{cases}; h(x) = \begin{cases} \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

1. Show that f is continuous at 0

   Proof. Want to show that for each $\epsilon > 0$, there exists $\delta > 0$ such that

   $$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \epsilon$$

   Let $\delta^2 = \epsilon$. Suppose $|x| < \delta$. Then

   $$|f(x) - f(0)| = |x^2 - x - 0| = |x^2 - x|$$

   From our assumption

   $$|x^2 - x| < |x^2| < \delta^2 = \epsilon$$

   Thus f is continuous at 0

   POG

2. Show that g is differentiable at 1

   Proof. Want to show that

   $$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

   exists and is finite. Let's first consider the limit when $x \in \mathbb{Q}$.

   $$\lim_{x\to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x\to 1} x + 2 = 3$$

   Now consider the limit when $x \notin \mathbb{Q}$.

   $$\lim_{x\to 1} \frac{3x - 1 - 2}{x - 1} = \lim_{x\to 1} \frac{3(x - 1)}{x - 1)} = 3$$

   Thus

   $$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = 3$$

   So it is differentiable at $x = 1$.

   POG

3. Show that $\lim_{x\to 0} h(x)$ does not exist.

Proof. Let $a(x) = \frac{1}{x^2}$. $\lim_{x\to 0} a(x) = \infty$. Then

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \sin(a(x))$$

Which is equivalent to

$$\lim_{x\to\infty} \sin x$$

5

which does not exist. Thus the $\lim_{x\to 0} h(x)$ does not exist.

POG

2 Problem 2 7 / 10

Part 1

✓ + **1 pts** **Correct strategy: aimed to show that $\lim_{x\to 0} f(x) = f(0)$**

+ **2 pts** Fully correct proof (e.g., squeeze theorem)

✓ + **1 pts** **Partialy correct proof**

Part 2

✓ + **1 pts** **Correct strategy: aimed to show that $\lim_{x\to 0} \frac{g(x)-g(1)}{x-1}$ existed**

+ **2 pts** Fully correct proof

+ **1 pts** Partially correct proof

✓ + **1 pts** **Claimed that $\lim_{x\in \mathbb{Q}} = \lim_{x\in\mathbb{I}}$ was sufficient for existence of ordinary limit (this is true but not obvious)**

Part 3

✓ + **2 pts** **Correct strategy: demonstrated there is sequence(s) of points in the domain with $x\to 0$ yet tending towards different output values**

+ **2 pts** Fully correct proof

✓ + **1 pts** **Partially correct proof**

4 This inequality is not true for x negative

5 Not rigorous

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3 Question 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable everywhere and such that

$$\lim_{x \to \infty} f'(x) = \infty$$

Show that f is not uniformly continuous

Proof. We want to show that there exists an $\epsilon > 0$ such that for all $\delta > 0$, and for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| > \epsilon$.
Let $\epsilon = 1$, and $y = x + \frac{\delta}{2}$. Thus $|x - y| = \frac{\delta}{2} < \delta$. Because f is differentiable everywhere, it is also continuous everywhere, so we can use the mean value theorem. By the mean value theorem, there exists $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. Because $f'(x)$ is unbounded, we can find $f'(c) > \frac{2}{\delta}$ for large enough (x, y). Then

$$|f(y) - f(x)| = |f(x) - f(y)| = |f'(c)||x - y| > \frac{2}{\delta} \cdot \frac{\delta}{2} = 1 = \epsilon$$

Hence, there exists $|f(x) - f(y)| > 1$ for all $\delta > 0$. Thus f is not uniformly continuous.

POG

3 Problem 3 **10 / 10**

✓ **- 0 pts** Correct

- 7 pts Solved problem for specific example rather than in the general case

- 6 pts Non-rigorous use of infinities (e.g., took a limit in the MVT expression $$f(x)-f(y) = f'(c)(x-y)$$ or something similar)

- 8 pts Invalid proof strategy

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4 Question 4

Define function $f : [0, 1] \to \mathbb{R}$

$$f(x) = x^4$$

1. Compute the upper sum $U(f, P_n)$
   Each rectangle has a width of $\frac{1}{n}$. Because x^4 is monotonically increasing on $[0, 1]$, the upper sum can be calculated with a right hand Riemann sum, given by the expression.

$$\frac{1}{n}\sum_{k=1}^{n}(\frac{k}{n})^4 = \frac{1}{n^5}\sum_{k=1}^{n}k^4$$

$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30n^5}$$

$$= \frac{6n^5+15n^4+10n^3-n}{30n^5}$$

2. Compute the lower sum $L(f, P_n)$
   We can calculate the lower sum with a left hand Riemann sum.

$$\frac{1}{n^5}\sum_{k=0}^{n-1}k^4$$

   Because the $k = 0$ term is $0^4 = 0$, we can change the starting index to 1

$$= \frac{1}{n^5}\sum_{k=1}^{n-1}k^4$$

$$= \frac{1}{n^5}\left(\sum_{k=1}^{n}k^4 - n^4\right)$$

$$= \frac{6n^5+15n^4+10n^3-n}{30n^5} - \frac{1}{n}$$

$$= \frac{6n^5+15n^4+10n^3-n}{30n^5} - \frac{30n^4}{30n^5}$$

$$= \frac{6n^5-15n^4+10n^3-n}{30n^5}$$

3. Compute $\lim_{n\to\infty} U(f, P_n)$ and $\lim_{n\to\infty} L(f, P_n)$

$$\lim_{n\to\infty} U(f, P_n) = \frac{6n^5+15n^4+10n^3-n}{30n^5} = \frac{1}{5}$$

$$\lim_{n\to\infty} L(f, P_n) = \frac{6n^5-15n^4+10n^3-n}{30n^5} = \frac{1}{5}$$

   By limit theorems.

4. Show that f is integrable on $[0,1]$ and we have

$$\int_0^1 f(x)dx = \frac{1}{5}$$

Proof. From part 3,

$$\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = \frac{1}{5} - \frac{1}{5} = 0$$

Thus f is integrable and

$$\int_0^1 f(x)dx = U(f,P_n) = L(f,P_n) = \frac{1}{5}$$

POG

4 Problem 4 **10 / 10**

Parts 1, 2

- **- 1 pts** Incorrect $$M(f, [\frac{k-1}{n}, \frac{k}{n}])$$
- **- 1 pts** Incorrect $$m(f, [\frac{k-1}{n}, \frac{k}{n}])$$
- **- 2 pts** Incorrect $$U(f, P_n)$$ with incorrect definition
- **- 1 pts** Incorrect $$U(f,P_n)$$ but correct definition
- **- 2 pts** Incorrect $$L(f, P_n)$$ with incorrect definition
- **- 1 pts** Incorrect $$L(f, P_n)$$ but correct definition

Part 3

- **- 1 pts** Incorrect $$\lim_{n\to\infty}U(f, P_n)$$
- **- 1 pts** Incorrect $$\lim_{n\to\infty}L(f, P_n)$$

Part 4

- **- 1 pts** Not showing $$\lim_{n\to\infty}(U(f,P_n)-L(f,P_n))=0$$
- **- 1 pts** Insufficient / unclear argument to conclude

✓ **- 0 pts** Correct

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5 Question 5

1. Show that for all $x, y \in [0,1]$ we have $|f(x) - f(y)| \leq |x - y|$.

   Proof. Assume without loss of generality that $x > y$. Because $\cos x$ is differentiable on $(0,1)$ and continuous on $[0,1]$ we can use the mean value theorem. By the mean value theorem, there exists $c \in (y, x)$ such that $\cos x - \cos y = f'(c)(x - y)$. This can be rearranged to

   $$\left|\frac{\cos x - \cos y}{x - y}\right| = |f'(c)|$$

   The derivative of $\cos x$ is $-\sin x$. So

   $$|f'(c)| = |\sin c| \leq 1$$

   Thus

   $$\left|\frac{\cos x - \cos y}{x - y}\right| \leq 1$$

   Multiply the $|x - y|$ to both sides

   $$|\cos x - \cos y| \leq |x - y|$$

   POG

2. Show that for all $k \in \{1, 2, \ldots, n\}$ we have

   $$M\left(f, \left[\frac{k-1}{n}, \frac{k}{n}\right]\right) - m\left(f, \left[\frac{k-1}{n}, \frac{k}{n}\right]\right) \leq \frac{1}{n}$$

   Proof. $\cos x$ is decreasing and nonnegative on $[0, \frac{\pi}{2}]$. Because $[0,1] \subset [0, \frac{\pi}{2}]$ it is also decreasing and nonnegative on $[0,1]$. Thus for all k

   $$M\left(f, \left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \cos\frac{k-1}{n}$$

   $$m\left(f, \left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \cos\frac{k}{n}$$

   So we want to show

   $$\cos\frac{k-1}{n} - \cos\frac{k}{n} \leq \frac{1}{n}$$

   From part 1,

   $$\left|\cos\frac{k-1}{n} - \cos\frac{k}{n}\right| \leq \left|\frac{k-1}{n} - \frac{k}{n}\right| = \left|\frac{1}{n}\right|$$

   Thus

   $$\cos\frac{k-1}{n} - \cos\frac{k}{n} \leq \frac{1}{n}$$

   POG

3. Show that $U(f, P_n) - L(f, P_n) \leq \frac{1}{n}$

Proof.

$$U(f, P_n) - L(f, P_n) = \frac{1}{n}\sum_{k=1}^{n} M\left(f, \left[\frac{k-1}{n}, \frac{k}{n}\right]\right) - \frac{1}{n}\sum_{k=1}^{n} m\left(f, \left[\frac{k-1}{n}, \frac{k}{n}\right]\right)$$

$$= \frac{1}{n}\left(\sum_{k=1}^{n} \cos\frac{k-1}{n} - \sum_{k=1}^{n} \cos\frac{k}{n}\right)$$

$$= \frac{1}{n}(\cos 0 - \cos 1 + \cos 1 - \cos 2 + \cdots + \cos\frac{n-1}{n} - \cos\frac{n}{n})$$

$$= \frac{1}{n}(\cos 0 - \cos 1) \leq \frac{1}{n}$$

Because $\cos 0 - \cos 1 = 1 - \cos 1 \leq 1$

POG

4. Show that f is integrable on $[0, 1]$

Proof. f is integrable $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$. From part 3,

$$0 \leq U(f, P_n) - L(f, P_n) \leq \frac{1}{n}$$

If we find the limit of both sides

$$\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) \leq \lim_{n\to\infty}\frac{1}{n} = 0$$

Thus $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n))$ is bounded above by 0. Furthermore, $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n))$ is bounded below by 0 because $U(f, P_n) \geq L(f, P_n)$. So by the squeeze theorem, $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$. Thus f is integrable.

POG

5 Problem 5 **10 / 10**

Part 1

- **1 pts** Not correctly apply the MVT
- **1 pts** Incorrect argument to conclude

Part 2

- **1 pts** Not correctly apply the EVT to find $$x_k,y_k\in [\frac{k-1}{n},\frac{k}{n}]$$ such that $$f(x_k)=M, f(y_k)=m$$, or something similar
- **1 pts** Not correctly apply Part 1
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Part 3

- **1 pts** Incorrect formula for $$U,L$$
- **2 pts** Incorrect argument to conclude
- **1 pts** Minor mistake

Part 4

- **1 pts** Not show $$\lim_{n\to\infty}(U(f, P_n)-L(f,P_n))=0$$ or something similar
- **1 pts** Incorrect argument to conclude

✓ **- 0 pts** Correct

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6 Question 6

1. $f(x) = |x|$

$$\lim_{h\to 0} \frac{f(h^2) - f(0)}{h^2} = \lim_{h\to 0} \frac{|h^2|}{h^2}$$

Because $|h^2| = h^2$

$$= \lim_{h\to 0} 1 = 1$$

However, f is not differentiable at 0

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

$$-1 \neq 1$$

Thus f is not differentiable

2. $g = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$

First we prove g is not integrable.

Proof. For any partition $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ we have

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = 1$$

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} -1 \cdot (t_k - t_{k-1}) = -1$$

The upper and lower Darboux integrals do not agree, thus g is not integrable. POG

Now we prove $g^2 = \begin{cases} 1 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases} = 1$ is integrable.

Proof.

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = 1$$

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = 1$$

The upper and lower Darboux integrals agree, thus g^2 is integrable. POG

3. $h = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$

First we prove h is discontinuous.

Proof. Let $a \in \mathbb{R}$. If $a \in \mathbb{Q}$, then there exists a sequence (x_n) of irrational numbers by the denseness of irrational numbers such that $\lim x_n = a$. Then $\lim h(x_n) = -1 \neq 1 = h(a)$. Similarly, if $a \notin \mathbb{Q}$, then there exists a sequence (r_n) of rational numbers by the denseness of rational numbers such that $\lim r_n = a$. Then $\lim h(r_n) = 1 \neq -1 = h(a)$. Thus h is discontinuous everywhere. POG

Now we prove h^2 is differentiable everywhere. $h^2 = \begin{cases} 1 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases} = 1$

Proof. Let $a \in \mathbb{R}$. For h^2 to be differentiable anywhere then,

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

must exist and be finite. Because $h = 1$

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{1 - 1}{x - a} = 0$$

Thus h is differentiable everywhere.

POG

6 Problem 6 **10 / 10**

Part 1

- **- 1 pts** $$\lim_{h\to 0}\frac{f(h^2)-f(0)}{h^2}$$ does not exist
- **- 1 pts** $$f$$ is differentiable at $$0$$
- **- 1 pts** Incorrect/insufficient/unclear proof

Part 2

- **- 1 pts** $$g$$ is integrable / does not have the domain $$[0,1]$$
- **- 1 pts** $$g^2$$ is not integrable
- **- 1 pts** Incorrect/insufficient/unclear proof

Part 3

- **- 1 pts** $$h$$ is continuous at some $$x\in \R$$
- **- 1 pts** $$h^2$$ is not differentiable at some $$x\in \R$$
- **- 1 pts** Incorrect/insufficient/unclear proof for discontinuity of $$h$$
- **- 1 pts** Incorrect/insufficient/unclear proof for differentiability of $$h^2$$

✓ **- 0 pts** Correct

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