Math 131A Midterm 2 Winter 2021

For the instructions for this test, see the dedicated PDF. In particular, remember that you have to show work for all of your steps/claims.

Please refer to the dedicated CCLE announcement for the times when we are available by email. Notice that only clarifications on the wording are allowed, while no hints nor feedback on your work will be provided.

1. **Copy** the statement in italic here below, **and sign it**.

   I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

   WARNING: per departmental policy, if you do not copy and **sign** this statement, you will be given a failing grade.
   DIRECTION: upload the signed statement as Question 1 on Gradescope.

2. **(9 pts)** Consider the function $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x-1}{x}$. **Using the $\epsilon-\delta$ definition of continuity** (no credit awarded otherwise!) prove that f is continuous on its domain.

3. (a) **(5 pts)** Assume that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ both converge. Further assume that, for every $n \in \mathbb{N}$, $x_n \leq y_n$. Prove that $\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$. (Hint: use the definition of sum of a series together with limit laws and problem 8 from homework 3).

   (b) **(9 pts)** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers. Assume that $\sum_{n=1}^{\infty} a_n$ converges to a real number $L \geq 0$. Prove that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges to a number M satisfying $0 \leq M \leq \dfrac{L}{5}$. (Hint: construct a suitable subsequence inductively, then apply part (a)). **Note:** you can use part (a) even if you did not prove it.

4. **(8 pts)** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous and increasing function. Assume that $f(0) = 1$. Prove that there exists $x \in [-1, 0)$ such that $f(x) = -x$.

5. **(9 pts)** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that, for every $n \in \mathbb{N}$, $|a_{n+1} - a_n| \leq \dfrac{1}{n^2}$. Show that $(a_n)_{n\in\mathbb{N}}$ is convergent.

6. **(5 pts)** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $\limsup a_n = 8$. Show that there exists $N \in \mathbb{N}$, such that for every $n \geq N$, $a_n \leq 9$.