Math 131A-1 Midterm 1 Spring 2020

Name: ______________________ uid: ______________________

Section: __________ Signature: ____________________________

Instructions:

- Unless otherwise stated, you need to justify your answer. Please show all of your work, as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown.
- You can use any theorem seen in class or assigned in the weekly homework problem set, as long as you state it clearly and correctly.
- You have 90 minutes to complete the exam **and** upload it on Gradescope.
- All answers should be completely simplified, unless otherwise stated.
- You can use your notes, your textbook, your homework, and all the material uploaded on the CCLE webpage.
- No resource other than the ones in the previous item is allowed on this test. Furthermore, collaboration is prohibited on the test. Any deviation from these rules will be considered cheating.

Problem	Points	Score
1	12	
2	10	
3	11	
4	8	
5	7	
6	2	
Total	50	

1. Write the following statements in mathematical language:

   (a) (3 pts) every natural number is divisible by 14 only if it is divisible by 7;

   (b) (3 pts) the negation of the sentence in part (a) (your answer **cannot** start with $\neg$);

   (c) (3 pts) some but not all natural numbers are even

   (d) (3 pts) the negation of the sentence in part (c) (your answer **cannot** start with $\neg$).

2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

   (a) (4 pts) Show that, if $\{a_n\}_{n\in\mathbb{N}}$ is bounded, then so is the sequence $\{a_n^2+2a_n+4\}_{n\in\mathbb{N}}$;

   (b) (6 pts) now assume that $\{a_n\}_{n\in\mathbb{N}}$ converges to 2. Prove **using the definition of convergence, without making use of limit laws**, that $\{a_n^3\}_{n\in\mathbb{N}}$ converges to 8. Hint: $|a_n^3-8| = |a_n-2||a_n^2+2a_n+4|$. You can use part (a) even if you did not do it.

3. Consider the interval $[0,\pi]$. Also, we write $[0,\pi]\cap\mathbb{Q}$ to denote the set of rational numbers contained in $[0,\pi]$, that is

$$[0,\pi]\cap\mathbb{Q} = \{x\in\mathbb{Q} | 0\le x\le\pi\}.$$

   (a) (2 pts) prove that π is an upper bound for $[0,\pi]\cap\mathbb{Q}$;

   (b) (6 pts) prove that $\pi = \sup([0,\pi]\cap\mathbb{Q})$;

   (c) (3 pts) provide an example of a set $S\subseteq\mathbb{R}$ so that $S\cap\mathbb{Q}\neq\emptyset$ and $\sup(S\cap\mathbb{Q})\neq \sup(S)$. You have to prove that the example you provide has the claimed properties.

4. (8 pts) Prove the following statement. For every positive real number x and every natural number $n\in\mathbb{N}$, we have $(4+3x)^n > 4^n$.

5. (7 pts) Prove **using the definition of convergence, without making use of limit laws**, that the sequence $\{a_n\}_{n\in\mathbb{N}}$ given by $a_n = \dfrac{5n^2-7n+1}{12n^2-7n+3}$ converges to $\frac{5}{12}$.

6. (2 pts) Let $\mathcal{C}_{\mathbb{Q}}$ denote the set of Cauchy sequences of rational numbers, and let $\sim$ denote the equivalence relation so that $\mathbb{R} = \mathcal{C}_{\mathbb{Q}}/\sim$. Let $\{a_n\}_{n\in\mathbb{N}}$ be an element of $\mathcal{C}_{\mathbb{Q}}$ that does not converge to 0. Find a Cauchy sequence of rational numbers $\{b_n\}_{n\in\mathbb{N}}$ so that $\{a_nb_n\}_{n\in\mathbb{N}}\sim\{c_n\}_{n\in\mathbb{N}}$, where $c_n = 1$ for every $n\in\mathbb{N}$. You need to prove all your claims.