Math 131A Final Winter 2021

**For the instructions for this test, see the dedicated PDF. In particular, remember that you have to show work for all of your steps/claims.**

**Please refer to the dedicated CCLE announcement for the times when we are available by email. Notice that only clarifications on the wording are allowed, while no hints nor feedback on your work will be provided.**

1. **Copy** the statement in italic here below, **and sign it**.

   *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

   **WARNING:** per departmental policy, if you do not copy and **sign** this statement, you will be given a failing grade.
   **DIRECTION:** upload the signed statement as Question 1 on Gradescope.

2. Consider the following functions from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ x^2 & \text{if } x \geq 0 \text{ and } x \in \mathbb{Q} \\ x & \text{if } x < 0 \text{ and } x \in \mathbb{Q} \end{cases}, \quad g(x) = \begin{cases} x^2 - 1 & \text{if } x \in \mathbb{Q} \\ 2x - 2 & \text{if } x \notin \mathbb{Q} \end{cases},$$

$$h(x) = \begin{cases} \cos\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   (a) **(5 pts)** Prove that $f$ is continuous at 0.

   (b) **(5 pts)** Prove that $g$ is differentiable at 1.

   (c) **(4 pts)** Prove that $\lim_{x \to 0} h(x)$ does not exist.

3. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be three bounded sequences of real numbers. Assume that the following holds:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, a_n - \epsilon \leq b_n \leq c_n + \epsilon.$$

   Further assume that $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are convergent with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$.

   (a) **(5 pts)** Prove that $\limsup b_n \leq \limsup c_n$. (Hint: you can apply exercise 9.9.(c) in Ross to suitable sequences)

   (b) **(3 pts)** Prove that $(b_n)_{n\in\mathbb{N}}$ is convergent and that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. (Hint: you can use part (a), even if you did not do it, and you can also use the fact that $\liminf a_n \leq \liminf b_n$)

4. **(8 pts)** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty} a_n = 0$. Prove that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that, for every $k \in \mathbb{N}$, $a_{n_k} \leq \frac{1}{3k}$.

5. **(5 pts)** Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Further assume that, for every $x \in \mathbb{R}$, $f(x) \in \mathbb{Q}$. Prove that $f$ is a constant function.

6. Given $n \in \mathbb{N}$, let $P_n$ denote the even partition into $n$ subintervals of the interval $[0,1]$. Consider

$$\begin{aligned} f \colon [0,1] &\to \mathbb{R} \\ x &\mapsto x^4 \end{aligned}$$

In the following, use that for every $m \in \mathbb{N}$

$$\sum_{k=1}^{m} k^4 = \frac{m(m+1)(2m+1)(3m^2+3m-1)}{30}.$$

(a) **(2 pts)** Compute $U(f, P_n)$ (i.e., give an explicit formula depending on $n$).

(b) **(2 pts)** Compute $L(f, P_n)$ (i.e., give an explicit formula depending on $n$).

(c) **(4 pts)** Compute $\lim_{n\to\infty} U(f, P_n)$ and $\lim_{n\to\infty} L(f, P_n)$ (you can use all limit laws, but you need to write a complete argument using them; referring to the degrees of numerator and denominator is not considered a full argument).

(d) **(4 pts)** Only making use of what we learned in §32 in Ross (so, do not use that continuous functions are integrable), conclude that $f$ is integrable on $[0,1]$, and compute $\int_0^1 x^4 dx$. (you should use parts (a)-(c), and you cannot use the fundamental theorem of calculus to get to the answer -although you can use it to check your work-)

7. In this question, you can use that $\sin(x)$ is differentaible with derivative $\cos(x)$, and that both of these functions are bounded by 1.

(a) **(3 pts)** Show that, for every $x_1, x_2 \in \mathbb{R}$, $|\sin(x_1) - \sin(x_2)| \leq |x_1 - x_2|$.

(b) **(3 pts)** Show that, for every interval $[a,b]$, $M(\sin, [a,b]) - m(\sin, [a,b]) \leq (b-a)$. (you can use part (a) even if you did not do it)

(c) **(4 pts)** Let $P_n$ denote the even partition of $[0,1]$ into $n$ subintervals. Show that the following inequality holds: $U(\sin, P_n) - L(\sin, P_n) \leq \frac{1}{n}$. (you can use part (b) even if you did not do it)

(d) **(3 pts)** Explicitly determine a value $n_0$ such that $|U(\sin, P_{n_0}) - \int_0^1 \sin(x)dx| \leq 0.1$ and $|L(\sin, P_{n_0}) - \int_0^1 \sin(x)dx| \leq 0.1$. (you can use part (c) even if you did not do it)