Math 131a-3
Spring 2016
Midterm I
4/20/16
Time Limit: 50 minutes

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Discussion Section: 3A

This exam contains 7 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, cheat sheets, cell phones or any calculator on this exam. You are *not* allowed to use any material other than what has been provided by the proctor.

You are required to show your student ID at this exam. The following rules apply:

- **Do not begin** until instructed by the proctor.
- If you need more space, use the back of the pages; clearly indicate when you have done this. There are sheets of scratch paper attached at the back of the exam. **These pages will be ignored while grading your test.** Do not use them to show work on any of the problems, only for scratch work.
- **Mysterious or unsupported answers will not receive full credit.** Points will be taken off for a correct answer with no work shown. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

Problem	Points	Score
1	10	9
2	10	10
3	10	10
4	10	10
Total:	40	39

Do not write in the table to the right.

1. (a) (5 points) Let $a_n = \frac{2n+1}{n-1}$. Given an $\varepsilon > 0$, find a value N so that if $n > N$,

$$|a_n - 2| < \varepsilon$$

N = $\frac{3}{\varepsilon} + 1$

$|a_n - 2| = \left| \frac{2n+1-2n+2}{n-1} \right| = \left| \frac{3}{n-1} \right| < \varepsilon$

$\frac{3}{n-1} < \varepsilon$

$\frac{1}{n-1} < \frac{\varepsilon}{3}$

$n-1 < \frac{3}{\varepsilon}$

$n < \frac{3}{\varepsilon} + 1$

(b) (5 points) Use the limit theorems to compute the limit of the following sequence,

$$\frac{1}{3\cdot 4}, \frac{3}{4\cdot 5}, \frac{5}{5\cdot 6}, \frac{7}{6\cdot 7}, \ldots$$

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$S(n) = \frac{2n-1}{(n+1)(n+2)}$ $\frac{2n+4-5}{n+2}$

$\lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{n} = 0$ since when $n \to \infty$, $n+1 \to \infty$.

Similarly, $\lim_{n\to\infty} \frac{1}{n+2} = 0$

$\therefore \lim_{n\to\infty} S_n = \left(\lim_{n\to\infty} \frac{1}{n+1}\right)\left(\lim_{n\to\infty} \frac{2n+4-5}{n+2}\right)$ by product rule

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$= \left(\lim_{n\to\infty} \frac{1}{n+1}\right)\left[\left(\lim_{n\to\infty} 2\right) - 5 \lim_{n\to\infty} \frac{1}{n+2}\right]$ by sum rule and constant multiple rule

$= 0 \times [2-0]$

$= 0$

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2. (a) (5 points) Let $S, T \subseteq \mathbb{R}$ be two subsets of the real numbers and suppose $\sup S, \sup T$ exist. If T contains an element which is larger than every element of S is it true that $\sup T > \sup S$? Give a counter-example if your answer is no (i.e. give sets S, T which satisfy the conditions but for which $\sup S \geq \sup T$) or give a proof if your answer is yes.

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No.

Let $T = \{0\}$, $S = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.

Then $\sup T = 0$, and $\sup S = 0$. $\sup T = \sup S$.

$-\frac{1}{n} < 0$

0 is an upper bound of S because:

$-\frac{3}{5} < -\frac{1}{5}$

$\because n > 0 \ \forall n \in \mathbb{N}$, $n = (n^{-1} \cdot n) \cdot n = n^{-1} \cdot n^2 > 0$.

$n \cdot n > 0$, and therefore $n^{-1} > 0$ By trichotomy, $-n^{-1} < 0$.

Suppose $\sup S = u < 0$. Then $u > -\frac{1}{n}$ Let $u = \frac{-p}{q}$, $p, q \in \mathbb{N}$.

$-\frac{1}{n} < \frac{-p}{q} < -\frac{1}{q}$ $\therefore$ u can't be an upper bound.

(b) (5 points) Suppose that $S, T \subseteq \mathbb{R}$ are two non-empty subsets of real numbers for which $\sup S, \sup T$ exist. Recall that $S \cup T$ is the union of S and T, and it contains every element from S as well as every element from T. Show that $\sup(S \cup T)$ exists, and that $\sup(S \cup T) = \max(\sup S, \sup T)$.

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S, T are non-empty, and since sup S, sup T exist, S, T are bounded above

For an element $a \in S \cup T$, either $a \in S$ or $a \in T$.

If $a \in S$, then $a \leq \sup S$; if $a \in T$, then $a \leq \sup T$.

Therefore $\forall a \in S \cup T$, there is an upper bound M such that $a \leq M$.

By completeness theorem, since $S \cup T$ has an upper bound, it has a supremum.

$\therefore \sup(S \cup T)$ exists.

As supremum is the least upper bound, if $\sup S < \sup T$, then $\sup S$ can't be an upper bound of T, and thus can't be an upper bound of $S \cup T$, and $\sup T \geq t$, $t \in T$, $\sup T > \sup S \geq s$, $s \in S$. $\sup T$ can be an upper bound of $S \cup T$.

Similarly if $\sup T < \sup S$. $\therefore$ $\max(\sup S, \sup T)$ is an upper bound of $S \cup T$

$\max(\sup S, \sup T) \geq \sup(S \cup T)$.

(On the back continued)

$\sup(S \cup T) \geq a, \ a \in S \cup T$.

Therefore $\sup(S \cup T)$ has to be larger than any element in S as well as in T.

$\therefore \sup(S \cup T)$ is an upper bound for both S and T.

$\therefore \sup(S \cup T) \geq \sup(S), \quad \sup(S \cup T) \geq \sup(T)$.

$\therefore \sup(S \cup T) \geq \max(\sup(S), \sup(T))$.

Since $\sup(S \cup T) \leq \max(\sup(S), \sup(T))$ as proved before,

$$\sup(S \cup T) = \max(\sup(S), \sup(T))$$

3. (10 points) Suppose a_n and b_n are sequences and the limits $\lim a_n = L$, $\lim b_n = M$ both exist. If $L < M$, show that there are only finitely many n with $b_n < a_n$. (Hint: show that there is some N with $a_n < b_n$ if $n > N$.)

By constant multiple theorem, $-\lim b_n = -M$.

By addition rules, $\lim a_n - b_n = L - M$.

For some N,

$|a_n - b_n + M - L| < \varepsilon$ for $n > N$

Since $L < M$, $M - L > 0$ let $M-L$ be ε.

$|a_n - b_n + M - L| < M - L$.

$a_n - b_n + (M-L) < M - L$.

$(M-L) - (a_n - b_n) - (M-L) > 0$.

$-(a_n - b_n) > 0$

$a_n - b_n < 0$. for some $n > N$.

4. (10 points) Give a rigorous proof using only the axioms of $\mathbb{R}$ that the following subset of real numbers is not bounded above,

$$S = \{n^2 \mid n \in \mathbb{N}\}$$

Assume S is bounded above for sake of contradiction.

Then by completeness theorem, $\exists$ a $\sup(S) = s$ such that $s \geq n^2$, $\forall n \in \mathbb{N}$.

As s is the least upper bound, $s - 2n + 1$ is not an upper bound.

Since $n \in \mathbb{N}$, $n \geq 1$.
For $n = 1$, $2n = 2 > 1$.
If $2n > 1$, then $2(n+1) = 2n + 2 > 1 + 2 > 1$.
$\therefore 2n > 1$ by induction. $\therefore 2n - 1 > 0$.

$s - \frac{1}{2}$

✓

However as $s \geq n^2$, $s - 2n + 1 \geq n^2 - 2n + 1 = (n-1)^2$.

$(n-1)^2 \in S$. $\therefore s$ is not the least upper bound.

$\therefore$ $\sup(S)$ does not exist, and S is not bounded above.