Math 131A **Name:** ______________________

ID Number ______________________

Spring 2021
Midterm 2
5/19/21

This exam contains 9 pages (including this cover page) and 4 problems.

This is exam is open notes, book, and lecture videos. You may *not* use any other outside resources on the exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You may use theorems proved in class, unless the statement of that particular problem instructs otherwise. If you use a theorem proved in class you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (a) (4 points) We showed in class that if a sequence $\{a_n\}$ converges to some limit $L \in \mathbb{R}$, then any subsequence $\{a_{k_n}\}$ also converges to L.

   Is an analogous result true for lim inf? That is, if $\{a_n\}$ is a sequence with $\liminf a_n = L$ for some $L \in \mathbb{R}$, it is necessarily true that every subsequence $\{a_{k_n}\}$ also satisfies $\liminf a_{k_n} = L$? Prove your answer.

   (b) (6 points) Prove that if $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded sequences of nonnegative real numbers, then

$$\limsup_{n\to\infty}(a_n \cdot b_n) \leq (\limsup_{n\to\infty} a_n) \cdot (\limsup_{n\to\infty} b_n).$$

   (Here the notation $\cdot$ denotes as usual the operation of multiplication on the real numbers.)

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2. (a) (5 points) Determine the convergence or divergence of the series

$$\sum_{n\geq 1} \left(\sin\left((2n+1)^2 \pi/3 \right) \right)^n .$$

Hint: A certain one of the series tests from the book/lecture notes may be very helpful here.

(b) (5 points) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a bounded, real-valued function with $\lim_{x\to\infty} f(x) = 0$, then the series

$$\sum_{n\geq 1} (f(1) \cdot f(2) \cdots f(n))$$

converges. (That is, the n^{th} term of this series is $f(1)$ times $f(2)$ times $\cdots$ times $f(n)$).

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3. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} (x+2)^2(x-3), & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$$

(a) (6 points) Where is $f(x)$ continuous, and where is $f(x)$ discontinuous? Prove your answer.

(b) (4 points) Where is $f(x)$ differentiable, and where is $f(x)$ nondifferentiable? Prove your answer.

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4. (a) (5 points) Prove that the function $f(x) = x^2 \sin(x)$ is uniformly continuous on any open interval (a, b) with $a, b \in \mathbb{R}$ and $a < b$, but it is not uniformly continuous on all of $\mathbb{R}$.

   (b) (5 points) (Unrelated to part (a)). Let $f(x)$ be a differentiable function on $\mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$. Prove that for any $\epsilon > 0$, there is an $x_0 \in \mathbb{R}$ such that $|f'(x_0)| \leq \epsilon$. Hint: can you use the Mean Value Theorem?

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