Math 131A **Name:** ______________________

ID Number ______________________

Spring 2021
Midterm 1
4/21/21

This exam contains 8 pages (including this cover page) and 4 problems.

This is exam is open notes, book, and lecture videos. You may *not* use any other outside resources on the exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You may use theorems proved in class, unless the statement of that particular problem instructs otherwise. If you use a theorem proved in class you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. Let $\mathbb{N}\times\mathbb{N}$ denote the **cartesian product** of $\mathbb{N}$ with itself; that is, $\mathbb{N}\times\mathbb{N}$ is the set of all ordered pairs (m,n) where m and n are natural numbers. Define a relation $<$ on $\mathbb{N}\times\mathbb{N}$ by

$$\big((m,n)<(j,k)\big) \iff \big((m<j) \text{ or } (m=j \text{ and } n<k)\big).$$

(a) (3 points) Prove that $<$ is an order on $\mathbb{N}\times\mathbb{N}$ (you can refer to the definition of an **order** from the lecture notes.)

An order is a relation satisfying the properties of trichotomy and transitivity, and this is clearly a relation as it is a subset of ordered pairs $((a,b);(c,d))$ of $\mathbb{N}\times\mathbb{N}$ satisfying $(a,b)<(c,d)$.

Trichotomy: Suppose $(m,n),(j,k)\in\mathbb{N}\times\mathbb{N}$. We have the following cases:

1. $(m,n)<(j,k)$; then $m<j$ or $m=j$ and $n<k$; in either of these subcases it is not possible that $(m,n)\geq(j,k)$ by definition of $<$, since then we require either $m>j$ or $m=j$ and $n\geq k$, and $<$ is itself an order on $\mathbb{N}$ and itself satisfies trichotomy.

2. $(m,n)=(j,k)$; then $m=j$ and $n=k$; in either of these subcases it is not possible that $(m,n)<(j,k)$ or $(m,n)>(j,k)$, since then either $m<j$ or $n<k$ or $m>j$ or $n>k$, none of which can be satisfied since $<$ is itself an order on $\mathbb{N}$ and itself satisfies trichotomy.

3. $(m,n)>(j,k)$ is analogous to the subcase $(m,n)<(j,k)$.

Transitivity: Suppose $(m,n),(j,k),(p,q)\in\mathbb{N}\times\mathbb{N}$ with $(m,n)<(j,k)$ and $(j,k)<(p,q)$. Then we have two cases, either $m<j$ or $m=j$ and $n<k$, and similarly we have two cases for the other inequality; so there are four cases total:

1. $m<j$ and $j<p$, which using transitivity of $<$ on $\mathbb{N}$ implies $m<p$, so by the definition of $<$ on $\mathbb{N}\times\mathbb{N}$ we have $(m,n)<(p,q)$;

2. $m<j$ and $j=p$ and $k<q$, which similarly implies using transitivity again that $m<p$, so $(m,n)<(p,q)$;

3. $m=j$ and $n<k$ and $j<p$, which similarly implies that $m<p$, so $(m,n)<(p,q)$;

4. $m=j$ and $n<k$ and $j=p$ and $k<q$, which similarly implies $m=p$ and $n<q$, so $(m,n)<(p,q)$.

(b) (3 points) Prove that if S is any nonempty subset of $\mathbb{N}\times\mathbb{N}$, then S has a least element: that is, there exists $(m,n)\in S$ such that $(m,n)\leq(j,k)$ for any $(j,k)\in S$. You may use the result of Exercise 9 in Homework 1.

Let $A = \{x : \exists y((x, y) \in S)\}$; that is, A is the set of all first coordinates of pairs in S. Then A is nonempty since S is nonempty, so by Exercise 9 in Homework 1 it has a least element, call this x_0. Define $B := \{y : (x_0, y) \in S\}$; then since $x_0 \in A$ this is nonempty, so again by Exercise 9 it has a least element, call this y_0. Then (x_0, y_0) is the least element of S, since for any $(m, n) \in S$ we have $x_0 \leq m$ and if $x_0 = m$, then $y_0 \leq n$.

(c) (4 points) In this part of the problem, you will prove a variant of mathematical induction on $\mathbb{N} \times \mathbb{N}$.

Suppose that $P(m, n)$ is a logical statement depending on $(m, n) \in \mathbb{N} \times \mathbb{N}$. Prove that if the following three statements hold, then $P(m, n)$ holds for all $(m, n) \in \mathbb{N} \times \mathbb{N}$:

1. $P(1, 1)$ is true;
2. $\forall m \in \mathbb{N}\Big(\big[\forall n \in \mathbb{N}(P(m, n))\big] \implies P(m + 1, 1)\Big)$;
3. $\forall (m, n) \in \mathbb{N} \times \mathbb{N}\Big(P(m, n) \implies P(m, n + 1)\Big)$.

Here we will iteratively use usual induction on $\mathbb{N}$ to prove this statement, which is known as "lexicographic induction." Suppose the three properties above hold; we will show that this implies that $P(m, n)$ holds for all $(m, n) \in \mathbb{N} \times \mathbb{N}$.

Define $Q(n) := P(1, n)$. We claim $Q(n)$ holds for all n. Base case: $Q(1) = P(1, 1)$ which is assumed to hold by (1). Inductive step: $Q(n) \implies Q(n + 1)$ is equivalent to $P(1, n) \implies P(1, n + 1)$ which follows by (3). By usual induction in $\mathbb{N}$, $Q(n)$ holds for all n.

Define $R(m) := \forall n(P(m, n))$. We claim $R(m)$ holds for all m, and once we prove this we're done. Base case: $R(1) = \forall n(Q(n))$, which was just shown previously. Inductive step: $R(m) \implies R(m + 1)$.

To show the inductive step for $R(m)$, we need to do another induction (an induction within an induction....sounds like a title for a movie starring Leo DiCaprio). So suppose $R(m)$ holds. Then by (2), we have that $P(m + 1, 1)$ holds. Let $S(n) := P(m + 1, n)$; then by usual induction on $\mathbb{N}$, $S(n)$ holds for all n since the base case $n = 1$ is already assumed to be true and $S(n) \implies S(n + 1)$ follows by (3). It follows again by usual induction on $\mathbb{N}$ that $R(m + 1)$ holds.

Thus we have completed the inductive step for $R(m)$ and shown that $R(m)$ holds for all m, completing the proof.

2. In what follows let A, B be subsets of the real numbers.

(a) (2 points) Show directly from the definition of supremum that if $A \subset B$, then $\sup A \leq \sup B$.

Suppose toward a contradiction that $\sup B < \sup A$. Then if x is an upper bound for A, then by definition of sup we have $\sup B < x$, so $\sup(B)$ is not an upper bound for A. This means there exists $a \in A$ such that $a > \sup(B)$ and hence $a > b$ for all $b \in B$, contradicting $A \subset B$.

(b) (5 points) We will say that B is **dense** in A if for any $s \in A$ and for any real number $\epsilon > 0$, there exists $t \in B$ such that $|s - t| < \epsilon$.

Show that if A is dense in B and B is dense in A, and moreover if $\sup A \notin A$ and $\sup B \notin B$, then $\sup A = \sup B$.

As we've shown previously, there exist monotonic increasing sequences $\{a_n\}$ and $\{b_n\}$ converging to $\sup A$ and $\sup B$, respectively, and since $\sup A \notin A$ and $\sup B \notin B$, they can be chosen to be strictly monotonic increasing. By denseness, for every n there is $x_n \in B$ such that $|x_n - a_n| < 1/n$. This allows us to inductively choose a sequence $\{x_n\}$ of elements of B satisfying $|x_n - a_n| < 1/n$ for all n, and since $x_n - a_n \to 0$ it must converge to $\sup A$. It follows that there is $x_n > \sup A - \epsilon$ for any $\epsilon > 0$, and hence $\sup B \geq \sup A$. Reversing the roles of A and B leads to $\sup A \geq \sup B$.

(c) (3 points) Let C be the collection of all subsets of the real numbers (this set C is also known more commonly as the **power set** of $\mathbb{R}$). Define a relation $\sim$ on C by

$$(A \sim B) \iff (A \text{ is dense in } B).$$

Is $\sim$ is an equivalence relation on C? Prove your answer.

This is not an equivalence relation; for instance it fails reflexivity. There are plenty of counterexamples one could construct, even some finite counterexamples. One infinite counterexample would be the following: the rationals $\mathbb{Q}$ are dense in the integers $\mathbb{Z}$, but not conversely.

3. (a) (1 point) State the definition of a sequence diverging to $+\infty$.

A sequence $\{x_n\}_{n\geq 1}$ diverges to $+\infty$ if for all real numbers $M > 0$, there exists a natural number N_M so that whenever $n \geq N_M$, we have $x_n > M$.

(b) (2 points) Use induction to prove that if $a > 0$ is an positive real number, then there exists a natural number N_a such that for all $n \geq N_a$,

$$\log_2(n) < n^a.$$

Note: this says qualitatively that $\log(n)$ grows more slowly than any positive power of n.

Let's prove this first for $a \geq 1$. It suffices for this case to prove $a = 1$. We know that there is some value of n for which $\log_2(n) < n$; for example $n = 2$ works. If this is true for a given n, then $\log_2(n+1) - \log(n) = \log_2((n+1)/n) \leq \log_2(3/2) < 1$, but $(n+1) - n = 1$, so $\log_2(n+1) < n+1$, proving the inductive step. So we have shown $\log_2(n) < n$ for all n sufficiently large.

For any $\epsilon > 0$, we can also use the above to show that $\log_2(x) < x^{1+\epsilon}$ for all sufficiently large real numbers x, since if x is a real number between n and $n+1$, then $\log_2(x) < \log_2(n+1) < \log_2(n) + 1 < n+1 \leq n^{1+\epsilon} \leq x^{1+\epsilon}$ as long as x is sufficiently large.

Now let's prove this for $a < 1$. Take $\epsilon = 1/a$ in the above, and plug in $x = n^a$ in the above. We get $a(\log_2(n)) = log_2(n^a) < n^{a+1}$. Rearranging this yields $\log_2(n) < \frac{n}{a}n^a$, and for n sufficiently large this is $< n^a$. Done.

(c) (3 points) Prove that if $\{a_n\}_{n\geq 1}$ is a sequence diverging to $+\infty$, then the sequence

$$\{b_n\}_{n\geq 1}, \qquad b_n := \log_2(1 + |a_n|)/a_n$$

converges to 0.

Hint: this should be related to what you showed in part b) about $\log(n)$ growing more slowly than any positive power of n.

Take n sufficiently large so that $|a_n| > M$ where M is sufficiently large so that $\log_2(n) < n^{1/4}$. for $n \geq M$. As above, this implies that $\log_2(x) < x^{1/2}$ for all x sufficiently large. Thus for $|a_n|$ sufficiently large we have $\log_2(1 + |a_n)/2|a_n| < 1/|a_n|^{1/2}$. Since $|a_n|$ diverges to $+\infty$, we can make $|a_n|$ sufficiently large by requiring n sufficiently large. This yields the inequality

$$0 \leq \log_2(1 + |a_n|)/|a_n| < 2/|a_n|^{1/2}$$

for all n sufficiently large. Since $|a_n|$ diverges to infinity, the left and right hand sides both converge to 0 (indeed, take $|a_n|$ sufficiently large to make $2/|a_n|^{1/2}$ smaller than any $\epsilon > 0$), and the result follows by the Squeeze Theorem.

(d) (4 points) Prove that if $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are sequences such that

$$\frac{a_n}{b_n} \to 0,$$

then there exists a sequence $\{c_n\}_{n\geq 1}$ such that

$$\frac{a_n}{c_n} \to 0, \qquad \frac{c_n}{b_n} \to 0.$$

Hint: You are looking for a sequence $\{c_n\}_{n\geq 1}$ whose behavior is in a sense "in between" that of $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$. Is there a way to use logarithms and the reasoning from part c) to your advantage?

There are a number of ways to do this. The sequence $c_n = a_n(\log(1 + |b_n/a_n|))$ will work. Indeed, $a_n/c_n = 1/(\log(1 + |b_n/a_n|)) \to 0$ since $|b_n/a_n| \to \infty$. (If one wishes, one can prove carefully the steps $a_n/b_n \to 0 \implies |b_n/a_n| \to \infty \implies \log(1 + |b_n/a_n|) \to \infty \implies a_n/c_n \to 0$.) One also sees that $c_n/b_n = a_n(\log(1 + |b_n/a_n|))/b_n \to 0$ by the previous part.

4. Let us say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ satisfies **Property X** if the following holds:

   "There exists a real number $C \geq 0$ so that for every natural number $n \in \mathbb{N}$, there exists a natural number $N(n)$ so that whenever $m \geq N(n)$, then $|a_m| \leq C + n$."

   Restated using logical symbols, we can rewrite this as:

   $$\exists C \geq 0(\ \forall n \in \mathbb{N}\ (\ \exists N(n) \in \mathbb{N}\ [\ m \geq N(n) \implies |a_m| \leq C + n]\ )\ ).$$

   (a) (1 point) Find the negation of Property X, written in logical symbols.

   $\forall C \geq 0(\ \exists n \in \mathbb{N}\ (\ \forall N \in \mathbb{N}\ (\exists m \in \mathbb{N}[\ (m \geq N)\ \wedge\ |a_m| > C + n]\ ))\ ).$

   (b) (2 points) Write down the definition of a bounded sequence as stated in either the notes or textbook, and then convert this to a statement in logical symbols.

   We say that a sequence $\{a_n\}_{n\geq 1}$ is bounded if there exists $C \geq 0$ such that $|a_n| \leq C$ for all n. $\exists C \geq 0(\forall n(|a_n| \leq C))$.

   (c) (3 points) Prove that Property X is logically equivalent to the logical statement from part b).

   Property X $\implies$ bounded:

   $$\text{Property X} \implies \exists C \geq 0(\exists N \in \mathbb{N}\ [\ m \geq N \implies |a_m| \leq C + 1]\ )\ ).$$

   $$\implies \exists C \geq 0\ (\forall m\ [|a_m| \leq \max(|a_1|, \ldots, |a_N|, C + 1)]\ )\ ) \implies \text{bounded}.$$

   Bounded $\implies$ Property X:

   $$\exists C \geq 0(\forall n(|a_n| \leq C)) \implies \exists C \geq 0[\ m \geq 1 \implies |a_m| \leq C + 1] \implies \text{Property X}.$$

   (d) (4 points) We say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ satisfies **Property Y** if the following holds:

   "There exists a real number $C \geq 0$ such that for every $n \in \mathbb{N}$, there exists at least n many different values of k for which $|a_k| \leq C$."

   Find an example of a sequence that satisfies Property Y but not Property X (1 point). Next, prove that every sequence satisfying Property Y has a convergent subsequence (3 points).

Any unbounded sequence with a convergent subsequence will work; say $a_n = 1$ if n is odd and $a_n = n$ for n even.

To see that a sequence satisfying Property Y has a convergent subsequence, first inductively choose a bounded subsequence as follows. Since we know that every nonempty subset of the natural numbers has a least element, inductively define k_n as follows. Since there is at least one element satisfying $|a_k| \leq C$, Take k_1 to be the least element of $\{m : |a_m| \leq C\}$. Supposing we have constructed $k_1 < k_2 < \cdots < k_n$, define k_{n+1} to be the minimum of the set $\{m : |a_m| \leq C, m > k_{n-1}\}$. This set is nonempty because there are at least $k_{n-1} + 1$ many elements of the sequence bounded by C.

Thus we have shown that $\{a_n\}$ has a bounded subsequence, and by Bolzano-Weierstrass we may choose a further subsequence that converges; clearly this is also a subsequence of the original sequence $\{a_n\}$.