Math 131A **Name:** ____________________

ID Number ____________________

Spring 2021
Midterm 1
4/21/21

This exam contains 9 pages (including this cover page) and 4 problems.

This is exam is open notes, book, and lecture videos. You may *not* use any other outside resources on the exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You may use theorems proved in class, unless the statement of that particular problem instructs otherwise. If you use a theorem proved in class you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. Let $\mathbb{N}\times\mathbb{N}$ denote the **cartesian product** of $\mathbb{N}$ with itself; that is, $\mathbb{N}\times\mathbb{N}$ is the set of all ordered pairs (m,n) where m and n are natural numbers. Define a relation $<$ on $\mathbb{N}\times\mathbb{N}$ by

$$\big((m,n)<(j,k)\big) \iff \big((m<j) \text{ or } (m=j \text{ and } n<k)\big).$$

(a) (3 points) Prove that $<$ is an order on $\mathbb{N}\times\mathbb{N}$ (you can refer to the definition of an **order** from the lecture notes.)

(b) (3 points) Prove that if S is any nonempty subset of $\mathbb{N}\times\mathbb{N}$, then S has a least element: that is, there exists $(m,n)\in S$ such that $(m,n)\le(j,k)$ for any $(j,k)\in S$. You may use the result of Exercise 9 in Homework 1.

(c) (4 points) In this part of the problem, you will prove a variant of mathematical induction on $\mathbb{N}\times\mathbb{N}$.

Suppose that $P(m,n)$ is a logical statement depending on $(m,n)\in\mathbb{N}\times\mathbb{N}$. Prove that if the following three statements hold, then $P(m,n)$ holds for all $(m,n)\in\mathbb{N}\times\mathbb{N}$:

1. $P(1,1)$ is true;
2. $\forall m\in\mathbb{N}\Big(\big[\forall n\in\mathbb{N}(P(m,n))\big] \implies P(m+1,1)\Big)$;
3. $\forall (m,n)\in\mathbb{N}\times\mathbb{N}\Big(P(m,n) \implies P(m,n+1)\Big)$.

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2. In what follows let A, B be subsets of the real numbers.

   (a) (2 points) Show directly from the definition of supremum that if $A \subset B$, then $\sup A \leq \sup B$.

   (b) (5 points) We will say that B is **dense** in A if for any $s \in A$ and for any real number $\epsilon > 0$, there exists $t \in B$ such that $|s - t| < \epsilon$.

   Show that if A is dense in B and B is dense in A, and moreover if $\sup A \notin A$ and $\sup B \notin B$, then $\sup A = \sup B$.

   (c) (3 points) Let C be the collection of all subsets of the real numbers (this set C is also known more commonly as the **power set** of $\mathbb{R}$). Define a relation $\sim$ on C by

   $$(A \sim B) \iff (A \text{ is dense in } B).$$

   Is $\sim$ is an equivalence relation on C? Prove your answer.

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3. (a) (1 point) State the definition of a sequence diverging to $+\infty$.

   (b) (2 points) Use induction to prove that if $a > 0$ is an positive real number, then there exists a natural number N_a such that for all $n \geq N_a$,

$$\log_2(n) < n^a.$$

   Note: this says qualitatively that $\log(n)$ grows more slowly than any positive power of n.

   (c) (3 points) Prove that if $\{a_n\}_{n\geq 1}$ is a sequence diverging to $+\infty$, then the sequence

$$\{b_n\}_{n\geq 1}, \qquad b_n := \log_2(1 + |a_n|)/a_n$$

   converges to 0.

   Hint: this should be related to what you showed in part b) about $\log(n)$ growing more slowly than any positive power of n.

   (d) (4 points) Prove that if $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are sequences such that

$$\frac{a_n}{b_n} \to 0,$$

   then there exists a sequence $\{c_n\}_{n\geq 1}$ such that

$$\frac{a_n}{c_n} \to 0, \qquad \frac{c_n}{b_n} \to 0.$$

   Hint: You are looking for a sequence $\{c_n\}_{n\geq 1}$ whose behavior is in a sense "in between" that of $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$. Is there a way to use logarithms and the reasoning from part c) to your advantage?

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4. Let us say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ satisfies **Property X** if the following holds:

   "There exists a real number $C \geq 0$ so that for every natural number $n \in \mathbb{N}$, there exists a natural number $N(n)$ so that whenever $m \geq N(n)$, then $|a_m| \leq C + n$."

   Restated using logical symbols, we can rewrite this as:

$$\exists C \geq 0 (\ \forall n \in \mathbb{N}\ (\ \exists N(n) \in \mathbb{N}\ [\ m \geq N(n) \implies |a_m| \leq C + n]\ )\ ).$$

   (a) (1 point) Find the negation of Property X, written in logical symbols.

   (b) (2 points) Write down the definition of a bounded sequence as stated in either the notes or textbook, and then convert this to a statement in logical symbols.

   (c) (3 points) Prove that Property X is logically equivalent to the logical statement from part b).

   (d) (4 points) We say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ satisfies **Property Y** if the following holds:

   "There exists a real number $C \geq 0$ such that for every $n \in \mathbb{N}$, there exists at least n many different values of k for which $|a_k| \leq C$."

   Find an example of a sequence that satisfies Property Y but not Property X (1 point). Next, prove that every sequence satisfying Property Y has a convergent subsequence (3 points).

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