Math 131A **Name:** ______________________

ID Number ______________________

Winter 2020
Final Exam
03/17/20

This exam contains 16 pages (including this cover page) and 8 problems.

You may use your book and notes on this exam, but not any other resources

You are required to show your work on each problem on this exam. The following rules apply:

- You may use theorems proved in class, unless the statement of that particular problem instructs otherwise. If you use a theorem proved in class you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	0	
6	10	
7	10	
8	10	
Total:	70	

1. (a) (5 points) Use induction to show that for all positive integers n, we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

   (b) (5 points) Use induction to show that for all positive integers n, and for all $x \neq 1$, we have

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

   Check also that you may prove the same identity by multiplying both sides by $x - 1$ and canceling out terms (the left hand side should *telescope*).

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2. Prove the following using ϵ's.

   (a) (5 points) Suppose that $\{a_n\}_{n\geq 1}$ is a sequence diverging to $-\infty$. Show that $\{a_n^2\}_{n\geq 1}$ diverges to $+\infty$.

   (b) (5 points) If $a_n \to L_1$ and $b_n \to L_2$ where L_1 and L_2 are real numbers then $a_n + b_n \to L_1 + L_2$.

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3. (10 points) Suppose that $\{a_n\}_{n\geq 1}$ is a sequence taking on finitely many values in the open interval $(-1,1)$. Show that $\sum_{n\geq 1}(a_n)^n$ converges.

   (Hint: it is crucial here that we have the *open* interval $(-1,1)$ and not the closed interval $[-1,1]$. Also, if you get stuck it may help to try playing with some specific examples.)

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4. (a) (5 points) Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that if $L \in \mathbb{R}$ is such that there is no subsequence of $\{a_n\}_{n\geq 1}$ converging to L, then there exists some $\epsilon > 0$ such that only finitely many terms of the sequence $\{a_n\}_{n\geq 1}$ lie in the interval $(L-\epsilon, L+\epsilon)$.

   (b) (5 points) Let $\{a_n\}_{n\geq 1}$ be a sequence such that $\liminf a_n > 0$. Prove that there is no subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that the series $\sum_{n\geq 1} a_{k_n}$ converges.

   (**A few remarks**: Part (b) is not particularly related to part (a). Also, the strictness of the inequality $\liminf a_n > 0$ in part (b) is important.)

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5. Let $f : [0, 1] \to \mathbb{R}$ be differentiable such that $|f'(x)| < 1$ for every $x \in [0, 1]$. Show that f has at most one fixed point in $[0, 1]$, i.e. at most one $a \in [0, 1]$ such that $f(a) = a$.

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6. (a) (4 points) Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

   (b) (6 points) Suppose f is a function defined on all of $\mathbb{R}$ such that $f(n) = n^2$ for every $n \in \mathbb{N}$. Prove that f is not uniformly continuous on $\mathbb{R}$.

   (Hint: Notice that $(n+1)^2 - n^2 \to \infty$ as $n \to \infty$. Let $\epsilon = 1$ (say), and suppose that there is some $\delta > 0$ that works. Now choose an n sufficiently large so that $(n+1)^2 - n^2$ is large enough, and then attempt to cover the interval $[n, n+1]$ by δ-intervals.)

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7. (10 points) Compute the Maclaurin series (i.e. Taylor series centered at $c = 0$) for the function $f(x) = e^x$, and prove that it converges to f for all $x \in \mathbb{R}$. Be sure to carefully show all your steps.

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8. (10 points) Note: in what follows, when we use the word *integrable* we will mean Riemann-Darboux integrable.

   Let $a, b \in \mathbb{R}$ with $a < b$, and suppose that f is a function defined on $[a, b]$ such that f is integrable on $[a, b]$. Suppose g is a function defined on $[a, b]$ such that $g(x) = f(x)$ for all but finitely many $x \in [a, b]$. Show that:

   (i) g is integrable on $[a, b]$,

   (ii) $\int_a^b f = \int_a^b g$.

   (Hint: As a first step you could try to show this when g and f agree for all but *one* point in $[a, b]$ and then try to generalize the proof.)