MATH 131A, SPRING 2020: MIDTERM 1

Name: ____________________________

Student ID Number: ____________________________

Signature: ____________________________

Instructions:

- **There are 5 problems.**
- **In your solutions, you may use without proof any theorem proven in the book/notes/videos for the class (unless the problem itself asks to prove a certain theorem). State clearly (and correctly) when you are using a theorem.**
- **Unless otherwise stated, you need to justify your answer. Please show all of your work, as partial credit will be given where appropriate, and there may be no credit given for problems where there is no work shown.**
- **You have 24 hours to complete and upload the exam on Gradescope, by 11:59pm (PDT) of Monday, April 20.**
- **Any collaboration is prohibited on the test, and will be considered cheating.**

Problem	**Points**	**Score**
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1. (20 points) Prove the *Sum of Squares Formula*:

$$1^2 + 2^2 + \cdots + n^2 \; = \; \sum_{k=1}^{n} k^2 \; = \; \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Problem 2.

(1) (10 points) Show that if $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers, then the sequence $\left(s_n^2 + 2s_n + 4\right)_{n\in\mathbb{N}}$ is also bounded.
(2) (10 points) Prove using the definition (and without using limit laws) that the sequence $(s_n)_{n\in\mathbb{N}}$ given by $s_n = \frac{n}{n^2+1}$ converges to 0.

Problem 3.

(1) (6 points) Prove that $\sqrt{2}$ is an upper bound for the set of real numbers $\left[0, \sqrt{2}\right] \cap \mathbb{Q}$.
(2) (6 points) Prove that $\sqrt{2} = \sup\left(\left[0, \sqrt{2}\right] \cap \mathbb{Q}\right)$.
(3) (8 points) Provide an example of a set $S \subseteq \mathbb{R}$ so that $S \cap \mathbb{Q} \neq \emptyset$ and $\sup(S \cap \mathbb{Q}) \neq \sup(S)$.
You have to prove that the example you provide has the claimed properties.

Problem 4.

(1) (10 points) Give an example of a bounded sequence of real numbers which is not Cauchy. Justify your answer.
(2) (10 points) Let (s_n) be a sequence of real numbers such that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$. Prove that (s_n) is a Cauchy sequence.

Problem 5. (20 points) Prove that the set of irrational numbers is dense in $\mathbb{R}$. That is, show that for every $a, b \in \mathbb{R}$ such that $a < b$, there exists an irrational number $c \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < c < b$.