Midterm (Math 115B)

Friday, May 1st

Name:____

ID:____

Please include a signature to the following statement with your solutions:

"*I assert, on my honor, that I have not received assistance of any kind from any other person, or given assistance to any other person, while working on the midterm.*"

Signature:____

You will have 24 hours to complete this exam. It will be due at 12:00 am on Saturday, 5/02. Please upload your solutions to Gradescope before the deadline.

The exam is open book/open notes. You can use any materials including the notes, your homeworks and quizzes, and any *general* resources you can find on the internet. However you are not allowed to get help from any other person, *or to use any online resources that give you full solutions to these problems.* This includes talking to someone in person, talking to someone online, or asking for help on any sort of online forum. Also it includes asking specific questions about the questions on the midterm, OR more general questions about the material in the course.

Show your work for these problems, don't just give an answer. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may use any results proved in Math115A, the lectures or pre-lectures, the textbook or the homework sets, but please make it clear when you are doing so. You may still earn partial credit even if your final answer is incorrect.

Question	Points	Score
1	15	
2	10	
3	15	
4	10	
Total:	50	

Name:________________________________

1. [15 pts] Let F be any field, and let V be a vector space over F with $\dim V = 2$. Let $T : V \to V$ be a linear operator. Assume that there is *exactly one* one dimensional subspace $W \subseteq V$ which is T-invariant (so any other one-dimensional subspace $W' \subseteq V$ is *not* T-invariant).

Let $x \in W$ be any nonzero vector. As shown in class, x is an eigenvector of T. Let $T(x) = \lambda x$ for $\lambda \in F$.

(a) [5 pts] Prove that λ is the only eigenvalue of T, and that $E_\lambda = W$.

(b) [5 pts] Let $y \in V$ be any vector with $y \notin W$. Prove that $T(y) = bx + \lambda y$ for some *nonzero* $b \in F$. [*Hint: Since* $y \notin W = \operatorname{span}\{x\}$, $\{x, y\}$ *is a basis for* V, *so you know that* $T(y) = bx + cy$. *You need to prove that* $b \neq 0$ *and* $c = \lambda$.]

(c) [5 pts] Prove that there is a basis $\beta = \{v_1, v_2\}$ for V for which $[T]_\beta = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. [*Hint: Start with the basis* $\{x, y\}$ *from part (b), and modify it to define* β.]

Name:________________________________

2. [10 pts] Let V be a finite dimensional *real* vector space, and let $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ be an inner product on V. Let $T : V \to V$ be a linear operator satisfying $T^3 = I$. Define a function $\langle\ ,\ \rangle_T : V \times V \to \mathbb{R}$ by

$$\langle x, y\rangle_T = \langle x, y\rangle + \langle Tx, Ty\rangle + \langle T^2x, T^2y\rangle.$$

(a) [5 pts] Prove that $\langle\ ,\ \rangle_T$ is also an inner product on V.

(b) [5 pts] Treat V as an inner product space using $\langle\ ,\ \rangle_T$ (*NOT* $\langle\ ,\ \rangle$) as the inner product. Prove that T is orthogonal with respect to $\langle\ ,\ \rangle_T$ (that is, that $\|T(x)\|_T = \|x\|_T$ for all $x \in V$, where $\|x\|_T = \sqrt{\langle x, x, \rangle_T}$).

Name:______________________________

3. [15 pts] Consider the real vector space $V = \mathbb{R}^2$. For any ordered basis $\beta = \{x_1, x_2\}$ for $\mathbb{R}^2$, let $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2\}$ be the corresponding dual basis for $(\mathbb{R}^2)^*$ (defined by $\mathsf{f}_1(x_1) = 1, \mathsf{f}_1(x_2) = 0, \mathsf{f}_2(x_1) = 0$ and $\mathsf{f}_2(x_2) = 1$).

We can use any basis $\beta = \{x_1, x_2\}$ for $\mathbb{R}^2$ to construct an isomorphism $\Theta_\beta : \mathbb{R}^2 \to (\mathbb{R}^2)^*$, defined by by $\Theta_\beta(x_1) = \mathsf{f}_1$ and $\Theta_\beta(x_2) = \mathsf{f}_2$ (you don't need to prove that this is an isomorphism).

Let $\alpha = \{(1,0),(0,1)\}$ be the standard basis for $\mathbb{R}^2$. Then for any $(a,b) \in \mathbb{R}^2$ the function $\Theta_\alpha(a,b) \in (\mathbb{R}^2)^*$ is defined by the formula

$$[\Theta_\alpha(a,b)]\,(x,y) = ax + by$$

(you don't need to prove this).

(a) [7 pts] Find a basis β for $\mathbb{R}^2$ which is *not* the standard basis in either order (so $\beta \neq \{(1,0),(0,1)\}$ and $\beta \neq \{(0,1),(1,0)\}$) such that $\Theta_\alpha : \mathbb{R}^2 \to (\mathbb{R}^2)^*$ and $\Theta_\beta : \mathbb{R}^2 \to (\mathbb{R}^2)^*$ are the same function.

(b) [8 pts] Define an isomorphism $T : \mathbb{R}^2 \to (\mathbb{R}^2)^*$ by

$$[T(a,b)](x,y) = bx - ay.$$

That is for and $(a,b) \in \mathbb{R}^2$, $T(a,b) \in (\mathbb{R}^2)^*$ is the function $[T(a,b)](x,y) = bx - ay$ (you don't need to prove that T is an isomorphism). Prove that T does not correspond to a basis of $\mathbb{R}^2$, that is, prove that T is not in the form $T = \Theta_\gamma$ for any basis γ of $\mathbb{R}^2$. [*Hint: For any $v \in \mathbb{R}^2$, what is $[T(v)](v)$?*]

Part (a) shows that different bases can give the same isomorphism, and part (b) shows that not every isomorphism comes from a basis.

Name:________________________________

4. [10 pts] Let V be any finite dimensional real vector space, and let $T : V \to V$ be a linear operator. Assume that the minimal polynomial of T is

$$p(t) = t^5 - 2t^3 - 2t^2 - 3t - 2 = (t+1)^2(t-2)(t^2+1).$$

Consider the polynomial

$$g(t) = t^4 - 1 = (t+1)(t-1)(t^2+1)$$

so that $g(T) : V \to V$ is the linear operator $T^4 - I : V \to V$. Define

$$W = \mathsf{R}(g(T)) = \operatorname{im} g(T) = \{(T^4 - I)v \mid v \in V\} \subseteq V.$$

We showed in class that W is a T-invariant subspace of V, and so there is a linear operator $T_W : W \to W$ defined by $T_W(x) = x$ for $x \in W$.

Find, with proof, the minimal polynomial of T_W.

Note: You may assume without proof that polynomials can be factored uniquely *as a product of monic irreducible polynomials, see Theorem E.9 in appendix E.*