Math 115AH Midterm

1a. (Proof adapted from Hoffman & Kunze)

Recall that $\mathcal{L}(V,V)$, the vector space consisting of all mappings from V to V, has dimension $(\dim(V))^2$; this was shown in Homework. Denote $n = \dim V$. Then, if we have n^2+1 mappings from V to V, they are necessarily linear dependent. Thus take $N = n^2+1$, so there must exist non trivial linear combination of $\{1, T, \ldots, T^{n^2}\}$ equal to the 0 vector in this vector space, or the 0 linear operator. We let these coefficients be our a_i's.

b. Note that part (a) showed the existence of some polynomial f such that $f(T) = 0$. f has degree at most n^2. Then, as degrees of polynomials are integers, there must exist some polynomial q st $q(T) = 0$ of minimal degree. Note that q can always be monic, as if q has degree d, $a_d \neq 0$, and $q'(T) = (a_d^{-1})\, q(T) = 0$, we can just take q to be q'.

We show that 1) all $g \in F[t]$ st $g(T) = 0$ satisfies $g = qh$ for $h \in F[t]$, and

2) q is unique.

1) Division algorithm tells us that all g admits $g = qh + r$, where $h, r \in F[t]$ and r has degree less than q, or is 0. If $g = qh + r$, then $g(T) = q(T)\,h(T) + r(T)$. Recall $g(T) = q(T) = 0$ so this is the same as $r(T) = 0$. Suppose that $r \neq 0$. Then, r is some ~~function~~ polynomial where $r(T) = 0$ with degree less than q. However, by construction, q is of minimal degree, so $r \neq 0$ would contradict the minimality of degree of q. Thus $r = 0$, and $g = qh$ simply.

2) Suppose that q_1 and q_2 are both polynomials of minimal degree and are monic st $q_1(T) = q_2(T) = 0$. Then, by Division Algorithm, $\exists\, h, r \in F[t]$ st $q_1 = hq_2 + r$, with r having degree less than q_2. By the same argument as above, $r \neq 0$ contradicts q_2 having minimal degree, so $q_1 = hq_2$. Note that q_1 and q_2 both have minimum degrees, so they have the same degree. Thus h must be a polynomial with degree 0 (as $h = 0$ wouldn't work either.) However, recall that q_1 and q_2 are both monic. Then, if a_d and b_d are the leading coefficients of q_1 and q_2, this would suggest $a_d = b_d = 1$ and $a_d = hb_d$, so $h = 1$. Thus $q_1 = 1 \cdot q_2$, or $q_1 = q_2$.

2. If $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are ON basis of $\mathbb{R}^3$ such that $\mathcal{B}$ has $\begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$ as its first vector, and $\mathcal{C}$'s first vector is perpendicular to the span of $(1,0,1)$ and $(1,2,1)$, and $\mathcal{D}$ has $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ and as its first vector, then:

$$[T]_S = [I]_{\mathcal{B},S} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} [I]_{S,\mathcal{B}},$$

$$[S]_S = [I]_{\mathcal{C},S} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [I]_{S,\mathcal{C}}, \quad \text{and} \quad [U]_S = [I]_{\mathcal{D},S} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} [I]_{S,\mathcal{D}}$$

These matrices are adapted from Lecture 13.

We use the following bases $\mathcal{B}, \mathcal{C}, \mathcal{D}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \begin{bmatrix} 0 \\ 3/\sqrt{13} \\ -2/\sqrt{13} \end{bmatrix}, \begin{bmatrix} -13/\sqrt{182} \\ 2/\sqrt{182} \\ 3/\sqrt{182} \end{bmatrix} \right\}.$$

Here, $\mathcal{B}_2$ was found by inspection to be some vector in $\mathbb{R}^3$ st $\mathcal{B}_1 \cdot \mathcal{B}_2 = 0$. Then, we chose $\mathcal{B}_3$ as $\mathcal{B}_1 \times \mathcal{B}_2$ scaled into a normal vector.

$$\mathcal{C} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

We chose $\mathcal{C}_1$ by crossing the given vectors together to get a vector $\perp$ to the mentioned plane. Then, we pick two orthonormal vectors that span the plane.

$$\mathcal{D} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

We choose $\mathcal{D}_2$ to be some vector st $\mathcal{D}_1 \cdot \mathcal{D}_2 = 0$. Then, take $\mathcal{D}_3$ to be $\mathcal{D}_1 \times \mathcal{D}_2$.

We also know that $[I]_{\mathcal{B},S}$ is simply the matrix with $\mathcal{B}$ as its columns, and the same holds for $[I]_{\mathcal{C},S}$ and $[I]_{\mathcal{D},S}$. Also, $[I]_{\mathcal{B},S}^{-1} = [I]_{S,\mathcal{B}}$, and we know from Problem 8 that $[I]_{\mathcal{B},S}^{-1}$ is $[I]_{\mathcal{B},S}^{T}$, as its columns are orthonormal. Again, the same holds for $\mathcal{C}$ and $\mathcal{D}$. So,

$$[U \circ S \circ T]_S = [I]_{\mathcal{D},S} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} [I]_{\mathcal{D},S}^{T} \cdot$$

$$[I]_{\mathcal{C},S} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [I]_{\mathcal{C},S}^{T} \cdot$$

$$[I]_{\mathcal{B},S} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} [I]_{\mathcal{B},S}^{T}.$$

2. To write it out as one long matrix, it's

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{14}} & 0 & -\frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{182}} \\ \frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ -\frac{13}{\sqrt{182}} & \frac{2}{\sqrt{182}} & \frac{3}{\sqrt{182}} \end{bmatrix}$$

Problem 3.

Let v, w, x all be arbitrary. Also, note that $d(v,w) := \|v-w\| = \sqrt{\langle v-w, v-w\rangle}$.

a. Recall that the ~~output of~~ $\sqrt{\ }$ ~~is~~ function is defined to be the "positive" square root, so that when there is a negative and positive root, the positive one is taken. Then $d(v,w) \geq 0$ by its definition. Note that $\sqrt{x} = 0$ iff $x = 0$, so $d(v,w) = 0 \iff \langle v-w, v-w\rangle = 0$. Recall by ~~defn~~ inner product properties, this holds iff $v - w = 0$, or $v = w$. Thus $d(v,w) = 0 \iff v = w$.

b. Since $d(v,w) = \sqrt{\langle v-w, v-w\rangle}$ and $d(w,v) = \sqrt{\langle w-v, w-v\rangle}$, it suffices to show that $\langle v-w, v-w\rangle = \langle w-v, w-v\rangle$. By linearity, we decompose LHS into

$$\langle v, v-w\rangle - \langle w, v-w\rangle = \langle v,v\rangle + \overline{(-1)}\langle v,w\rangle - \langle w,v\rangle - \overline{(-1)}\langle w,w\rangle$$

$$= \langle v,v\rangle - \langle v,w\rangle - \langle w,v\rangle + \langle w,w\rangle.$$

Similarly, applying linearity to RHS yields

$$\langle w, w-v\rangle - \langle v, w-v\rangle = \langle w,w\rangle + \overline{(-1)}\langle w,v\rangle - \langle v,w\rangle - \overline{(-1)}\langle v,v\rangle$$

$$= \langle w,w\rangle - \langle w,v\rangle - \langle v,w\rangle + \langle v,v\rangle$$

By associativity & commutativity of addition in fields, we know these two are equal, and thus

~~$\langle v, v-w$~~ $\langle v-w, v-w\rangle = \langle w-v, w-v\rangle \Rightarrow d(v,w) = d(w,v)$.

c. We'll use Cauchy-Schwarz, proved in lecture: $|\langle v_1, v_2\rangle| \leq \|v_1\|\|v_2\|$. Consider $d(v,w) = \|v-w\|$. Note that $v - w = v - x + x - w$, so $d(v,w) = \|v-x+x-w\|$. Take $(d(v,w))^2$:

$d(v,w)^2 = \langle v-x+x-w, v-x+x-w\rangle$, and we rewrite this with linearity of $\langle\ \rangle$:

$$= \langle v-x, v-x\rangle + \langle v-x, x-w\rangle + \langle x-w, v-x\rangle + \langle x-w, x-w\rangle$$

$$= d(v-x)^2 + d(x-w)^2 + \langle v-x, x-w\rangle + \langle x-w, v-x\rangle.$$

Since Cauchy-Schwarz implies $\langle v-x, x-w\rangle \leq |\langle v-x, x-w\rangle| \leq \|v-x\|\|x-w\|$, and similarly, $\langle x-w, v-x\rangle \leq |\langle x-w, v-x\rangle| \leq \|x-w\|\|v-x\|$,

we know $d(v,w)^2 \leq \|v-x\|^2 + \|x-w\|^2 + 2\|x-w\|\|v-x\| = (\|v-x\| + \|x-w\|)^2$.

~~As $d(v,w)^2 \geq 0$, we know this implies~~

As $d(v,w) \geq 0$, and the square function is monotonically increasing on positives, we know

$$\|v-w\|^2 \leq (\|v-x\| + \|x-w\|)^2 \Rightarrow \|v-w\| \leq \|v-x\| + \|x-w\|, \text{ or}$$

$$d(v,w) \leq d(v,x) + d(x,w),$$ as desired.

Problem 4:

For ease of grading, our ON set is: $\left\{ \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}x, \frac{\sqrt{10}(3x^2-1)}{4}, \frac{\sqrt{14}(5x^3-3x)}{4} \right\}$.

Derivation: 1) $\int_{-1}^{1} 1\,dx = 2 \Rightarrow \|1\| = \sqrt{2} \Rightarrow u_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, and v_1 was 1

2) $x_{\{\frac{\sqrt{2}}{2}\}} = \langle x, \frac{\sqrt{2}}{2} \rangle \frac{\sqrt{2}}{2} = 0 \Rightarrow v_2 = x - 0 = x$. However, $\|x\|^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$

$\Rightarrow u_2 = \frac{x}{\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}\,x}{\sqrt{2}} = \frac{x\sqrt{6}}{2}$.

3) $x^2_{\{\frac{\sqrt{2}}{2}, \frac{x\sqrt{6}}{2}\}} = \langle x^2, \frac{\sqrt{2}}{2} \rangle \frac{\sqrt{2}}{2} + \langle x^2, \frac{x\sqrt{6}}{2} \rangle \frac{x\sqrt{6}}{2} = \frac{1}{3} + 0 = \frac{1}{3}$. Then, $v_2 = x^2 - \frac{1}{3}$

$\|v_2\|^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2\,dx = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9}\,dx = \frac{8}{45} \Rightarrow \|v_2\| = \frac{2\sqrt{2}}{3\sqrt{5}}$

$u_2 = \frac{x^2 - 1/3}{2\sqrt{2}} \cdot 3\sqrt{5} = \frac{(3x^2-1)\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{10}(3x^2-1)}{4}$

4) $x^3_{\{\frac{\sqrt{2}}{2}, \frac{x\sqrt{6}}{2}, \frac{\sqrt{10}(3x^2-1)}{4}\}} = \langle x^3, \frac{\sqrt{2}}{2} \rangle \frac{\sqrt{2}}{2} + \langle x^3, \frac{x\sqrt{6}}{2} \rangle \frac{x\sqrt{6}}{2} + \langle x^3, \frac{\sqrt{10}}{4}(3x^2-1) \rangle \frac{\sqrt{10}}{4}(3x^2-1)$

$= 0 + \frac{\sqrt{6}}{2}\int_{-1}^{1} x^4\,dx \left(\frac{\sqrt{6}}{2}x\right) + 0 = \frac{3}{5}x$

$\Rightarrow V_4 = x^3 - \frac{3}{5}x$. $\|v_4\|^2 = \int_{-1}^{1} (x^3 - \frac{3}{5}x)^2\,dx = \int_{-1}^{1} x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2 = \frac{8}{175}$.

Then, $u_4 = \frac{v_4}{\|v_4\|} = \frac{x^3 - 3/5\,x}{\frac{2\sqrt{2}}{5\sqrt{7}}} = \frac{\sqrt{7}(5x^3 - 3x)}{2\sqrt{2}} = \frac{\sqrt{14}(5x^3-3x)}{4}$.

Note that many of our integrals during Gram-Schmidt disappear to 0 because our integration bounds are symmetric, so all odd functions have ~~negative~~ integral 0.

Problem 5.

Suppose that $v = \sum_{i=1}^{n} \alpha_i v_i$ and $w = \sum_{i=1}^{n} \beta_i v_i$. Then, $\langle v, w \rangle = \langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{n} \beta_j v_j \rangle$.

By linearity, this is equal to $\sum_{i=1}^{n} \alpha_i \langle v_i, \sum_{j=1}^{n} \beta_j v_j \rangle = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \bar{\beta_j} \langle v_i, v_j \rangle$. Distributing yields $\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\beta_j} \langle v_i, v_j \rangle$. However, as $v_1 \dots v_n$ were orthonormal, $\langle v_i, v_j \rangle = 0$ if $i \neq j$ and $\langle v_i, v_i \rangle = 1$. Thus $\langle v, w \rangle = \sum_{i=1}^{n} \alpha_i \bar{\beta_i} (1) = \sum_{i=1}^{n} \alpha_i \bar{\beta_i}$.

consider RHS of Parseval's : note that $\overline{\langle w, v_i \rangle} = \langle v_i, w \rangle$. Furthermore, $\langle v_i, w \rangle = \langle v_i, \sum_{j=1}^{n} \beta_j v_j \rangle$.

~~By the same logic as earlier, $\langle v_i, v_j \rangle = 0$ for $i \neq j$ implies this is equivalent to~~

Then, $\langle v_i, w \rangle = \sum_{j=1}^{n} \bar{\beta_j} \langle v_i, v_j \rangle$ by linearity, and by the same logic as earlier, $i \neq j \Rightarrow \langle v_i, v_j \rangle = 0$ and $\langle v_i, v_i \rangle = 1$, so this simply becomes $\bar{\beta_i}$. The other term under the summation is $\langle v, v_i \rangle$, which can be rewritten as $\langle \sum_{j=1}^{n} \alpha_j v_j, v_i \rangle$. Following the same process as for $\langle v_i, w \rangle$, we see that $\langle v_i, w \rangle = \sum_{j=1}^{n} \alpha_j \langle v_j, v_i \rangle = \alpha_i$. So, the RHS of Parseval's becomes:

$$\sum_{i=1}^{n} \langle v, v_i \rangle \overline{\langle w, v_i \rangle} = \sum_{i=1}^{n} \alpha_i \bar{\beta_i}.$$

But, this was equal to the expression we found equal to the LHS. Thus

$$\langle v, w \rangle = \sum_{i=1}^{n} \alpha_i \bar{\beta_i} = \sum_{i=1}^{n} \langle v, v_i \rangle \overline{\langle w, v_i \rangle}, \text{ as desired.}$$

To derive Pythagorean Theorem, let $w = v$. Then, the LHS is $\langle v, w \rangle = \langle v, v \rangle = \|v\|^2$. ~~Since~~ The RHS becomes $\sum_{i=1}^{n} \langle v, v_i \rangle \overline{\langle v, v_i \rangle}$. We use the property that in $\mathbb{R}$ and in $\mathbb{C}$, $\bar{z} z = |z|^2$ to show that the RHS is equal to $\sum_{i=1}^{n} |\langle v, v_i \rangle|^2$.

Putting it all together, Parseval's tells us that

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, v_i \rangle|^2.$$

Problem 6a.

We show that $i \Rightarrow ii \Rightarrow iii \Rightarrow iv \Rightarrow v \Rightarrow i$.

($i \Rightarrow ii$) Suppose T preserves inner products. Then, trivially, T also preserves ~~linear~~ inner products and it suffices to just show that T is an isomorphism. Since $\dim V = \dim W$, by isomorphism theorem, T is isomorphic $\Leftrightarrow$ T is injective. Suppose $T(v_1) = T(v_2)$. Then, $Tv_1 - Tv_2 = 0$, and by inner product properties: $\langle Tv_1 - Tv_2, Tv_1 - Tv_2 \rangle = 0$. Linearity shows that this is equivalent to $\langle Tv_1, Tv_1 \rangle + \langle Tv_2, Tv_2 \rangle - \langle Tv_2, Tv_1 \rangle - \langle Tv_1, Tv_2 \rangle = 0$.

Since T preserves inner products, the LHS above is equivalent to

$$\langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle - \langle v_2, v_1 \rangle - \langle v_1, v_2 \rangle, \text{ or also}$$

$$\langle v_1 - v_2, v_1 - v_2 \rangle.$$

Thus $\langle v_1 - v_2, v_1 - v_2 \rangle = 0$, which is true iff $v_1 - v_2 = 0$ by inner product properties, so $v_1 = v_2$ and T is injective, and thus isomorphic.

$ii \rightarrow iii$: We wish to show that if $v_1 \dots v_n$ is an ON basis for V, then $Tv_1 \dots Tv_n$ is an ON basis for W. Note that $Tv_1 \dots Tv_n$ is ON if ~~all~~ for all $i \neq j$, $\langle Tv_i, Tv_j \rangle = 0$, and $\langle Tv_i, Tv_i \rangle = 1$. ~~Note that~~ By assumption of (ii), T preserves inner products and thus $\langle Tv_i, Tv_j \rangle = \langle v_i, v_j \rangle$, and as $v_1 \dots v_n$ are ON, $\langle Tv_i, Tv_j \rangle = \langle v_i, v_j \rangle = 0$. Similarly, $\langle Tv_i, Tv_i \rangle = \langle v_i, v_i \rangle = 1$. Finally, we note that $Tv_1 \dots Tv_n$ is a collection of $n = \dim W$ vectors, so ~~we~~ it suffices to show $Tv_1 \dots Tv_n$ are linearly independent. Let $w_i := Tv_i$. Suppose $\sum_{i=1}^{n} \alpha_i w_i = 0$. Let $j \in \{1 \dots n\}$ be arbitrary. Then,

$$\left\langle \sum_{i=1}^{n} \alpha_i w_i, w_j \right\rangle = \sum_{i=1}^{n} \alpha_i \langle w_i, w_j \rangle = \alpha_j \langle w_j, w_j \rangle = \alpha_j = 0$$

by linearity and then orthogonality of w_i's. Thus $w_1 \dots w_n$ forms a linearly independent set of $\dim W$ vertices, and is an ON basis for W. (It might also be said that all of $w_i \neq 0$ because T is an isomorphism.)

(iii => iv) ($n = \dim V = \dim W$)

Suppose that $v_1 \dots v_n$ is some ON basis for V so that $Tv_1 \dots Tv_n$ is an ON basis for W. We wish to show that if $w_1 \dots w_n$ is an ON basis for W, then $\exists x_1 \dots x_n$ an ON basis for V, where $Tx_i = w_i$. First, we show T is an isomorphism:

By Isomorphism Theorem, since $\dim V = \dim W$, it suffices to show T is an epimorphism. Let $y \in W$ be arbitrary; we want to show $\exists x$ st $Tx = y$. Let $y = \sum_{i=1}^{n} \alpha_i Tv_i$, as $Tv_1 \dots Tv_n$ is basis for W. Then, let x be $\sum_{i=1}^{n} \alpha_i v_i$. By linearity of T, it follows that $Tx = \sum_{i=1}^{n} \alpha_i Tv_i = y$, as desired. So, T is an epimorphism => isomorphism.

Since T is an ~~epimorphism~~ isomorphism, T^{-1} exists. We show $T^{-1} w_i$ forms an ON set, so then $x_i := T^{-1} w_i$ satisfies our requirements. Let $i \neq j$ be arbitrary in $\{1 \dots n\}$. We want to show 1) $\langle T^{-1} w_i, T^{-1} w_i \rangle = 1$ and 2) $\langle T^{-1} w_i, T^{-1} w_j \rangle = 0$.

Let $T^{-1} w_i = \sum_{k=1}^{n} \alpha_k v_k$, and $T^{-1} w_j = \sum_{l=1}^{n} \beta_l v_l$. It follows $w_i = \sum_{k=1}^{n} \alpha_k Tv_k$, $w_j = \sum_{l=1}^{n} \beta_l Tv_l$.

1) Note that $\langle w_i, w_i \rangle = 1$, as w_i's form ON basis. Also, $\langle w_i, w_i \rangle = \langle \sum_{k=1}^{n} \alpha_k Tv_k, \sum_{k=1}^{n} \alpha_k Tv_k \rangle$

$= \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \bar{\alpha}_l \langle Tv_k, Tv_l \rangle$ by inner product linearity. Since $Tv_1 \dots Tv_n$ ON, we have $\langle Tv_k, Tv_l \rangle = \delta_{k,l}$ => $\langle w_i, w_i \rangle = \sum_{k=1}^{n} \alpha_k \bar{\alpha}_k = 1$.

Then, $\langle T^{-1} w_i, T^{-1} w_i \rangle = \langle \sum_{k=1}^{n} \alpha_k v_k, \sum_{l=1}^{n} \alpha_l v_l \rangle = \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \bar{\alpha}_l \langle v_k, v_l \rangle$.

Since $v_1 \dots v_n$ also ON, $= \sum_{k=1}^{n} \alpha_k \bar{\alpha}_l$, which we know is 1 from earlier.

2) Similarly, $\langle w_i, w_j \rangle = 0$. So, $\langle \sum_{k=1}^{n} \alpha_k Tv_k, \sum_{l=1}^{n} \beta_l Tv_l \rangle = \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \bar{\beta}_l \langle Tv_k, Tv_l \rangle$

$= \sum_{k=1}^{n} \alpha_k \bar{\beta}_k = 0$.

Then, ~~similarly~~ $\langle T^{-1} w_i, T^{-1} w_j \rangle = \langle \sum_{k=1}^{n} \alpha_k v_k, \sum_{l=1}^{n} \beta_l v_l \rangle = \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \bar{\beta}_l \langle v_k, v_l \rangle$

$= \sum_{k=1}^{n} \alpha_k \bar{\beta}_k = 0$, from before. Thus x_i and x_j are orthogonal.

We've showed $x_1 \dots x_n$ are orthogonal, so it suffices to show they're a basis. Note that these are $\dim V$ vectors in V, so it suffices to show they're linearly independent. Suppose ~~any~~ $\sum_{i=1}^{n} \alpha_i x_i = 0$. Let $x_j \in \{x_1 \dots x_n\}$ arbitrary. Then, $\langle \sum_{i=1}^{n} \alpha_i x_i, x_j \rangle = \langle 0, x_j \rangle = 0$.

=> $\sum_{i=1}^{n} \alpha_i \langle x_i, x_j \rangle = \alpha_j \langle x_j, x_j \rangle = \alpha_j = 0$ for all j. Then, $x_1 \dots x_n$ are ON and linearly independent => an ON basis.

(iv => v) let $w_1 \dots w_n$ be an ON basis for W, and $v_1 \dots v_n$ be an ON basis st $Tv_i = w_i$ for $i = 1 \dots n$. Consider arbitrary $v = \alpha_1 v_1 + \dots + \alpha_n v_n$.

Then, $\langle Tv, Tv \rangle = \langle T(\alpha_1 v_1 + \dots + \alpha_n v_n), T(\alpha_1 v_1 + \dots + \alpha_n v_n) \rangle$

$= \langle \sum_{i=1}^{n} \alpha_i Tv_i, \sum_{j=1}^{n} \alpha_j Tv_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha_j} \langle Tv_i, Tv_j \rangle$ by linearity of T and $\langle \cdot, \cdot \rangle$. Then, as $\langle Tv_i, Tv_j \rangle = \delta_{ij}$, this is equal to $\sum_{i=1}^{n} \alpha_i \bar{\alpha_i}$.

Similarly, $\langle v, v \rangle = \langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{n} \alpha_j v_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha_j} \langle v_i, v_j \rangle$. v_i's also form ON basis, so $\langle v, v \rangle = \sum_{i=1}^{n} \alpha_i \bar{\alpha_i}$.

Thus $\langle v, v \rangle = \sum_{i=1}^{n} \alpha_i \bar{\alpha_i} = \langle Tv, Tv \rangle$, so $||v|| = ||Tv||$, as $||x|| = \sqrt{\langle x, x \rangle}$.

(v => i) Suppose that $||Tv|| = ||v||$ for all v, so $\langle Tv, Tv \rangle = \langle v, v \rangle$ also. In particular, this holds for $v = v_1 + v_2$, where v_1, v_2 arbitrary. Then, $\langle Tv_1 + Tv_2, Tv_1 + Tv_2 \rangle = \langle v_1 + v_2, v_1 + v_2 \rangle$.
Expanding via linearity of inner product yields:

$$\langle Tv_1, Tv_1 \rangle + \langle Tv_1, Tv_2 \rangle + \langle Tv_2, Tv_1 \rangle + \langle Tv_2, Tv_2 \rangle = \langle v_1, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_1 \rangle + \langle v_2, v_2 \rangle$$

Since $||Tv_1|| = ||v_1||$ and $||Tv_2|| = ||v_2||$, we're left with:

$$\langle Tv_1, Tv_2 \rangle + \langle Tv_2, Tv_1 \rangle = \langle v_1, v_2 \rangle + \langle v_2, v_1 \rangle. \quad (*)$$

Note that if $\mathbb{F} = \mathbb{R}$, then $\langle x, y \rangle = \overline{\langle y, x \rangle} = \langle y, x \rangle$ implies $\langle Tv_1, Tv_2 \rangle = \langle v_1, v_2 \rangle$ after dividing (*) by 2. So, we'd be done. Suppose, then, $\mathbb{F} = \mathbb{C}$ instead.

Then, we also evaluate $v = v_1 + iv_2$, so $\langle Tv, Tv \rangle = \langle v, v \rangle$

$\Rightarrow \langle Tv_1 + iTv_2, Tv_1 + iTv_2 \rangle = \langle v_1 + iv_2, v_1 + iv_2 \rangle$.

expanding due to linearity of $\langle , \rangle$

$\Rightarrow \langle Tv_1, Tv_1 \rangle + i \langle Tv_2, Tv_1 \rangle + \bar{i} \langle Tv_1, Tv_2 \rangle + \langle Tv_2, Tv_2 \rangle = \langle v_1, v_1 \rangle + i \langle v_2, v_1 \rangle + \bar{i} \langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle$

Again, $||Tv_1|| = ||v_1||$ and $||Tv_2|| = ||v_2||$ allows us to cancel terms on both sides:

$$i \langle Tv_2, Tv_1 \rangle - i \langle Tv_1, Tv_2 \rangle = i \langle v_2, v_1 \rangle - i \langle v_1, v_2 \rangle$$

Divide by i:

$$\langle Tv_2, Tv_1 \rangle - \langle Tv_1, Tv_2 \rangle = \langle v_2, v_1 \rangle - \langle v_1, v_2 \rangle.$$

Subtract (*) from this equation to get

$$-2 \langle Tv_1, Tv_2 \rangle = -2 \langle v_1, v_2 \rangle$$

Dividing by -2 yields

$$\langle Tv_1, Tv_2 \rangle = \langle v_1, v_2 \rangle,$$ as desired.

6b.

Suppose that V_1, V_2 have the same dimension. We will define an isometry from V_1 to V_2. Let $v_1 \dots v_n$ be an ON basis for V_1 and $w_1 \dots w_n$ an ON basis for V_2. Define T to be the transformation that takes Tv_i to w_i for $i = 1 \dots n$, as it suffices to define linear transformation on the basis of the domain. We show that T is an isometry: let $v \in V_1$ be arbitrary, and $v = \sum_{i=1}^{n} \alpha_i v_i$. Then, $\langle v, v \rangle = \langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{n} \alpha_j v_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha_j} \langle v_i, v_j \rangle$

$= \sum_{i=1}^{n} \alpha_i \bar{\alpha_i}$, as $v_1 \dots v_n$ is an ON basis.

Similarly, $\langle Tv_i, Tv_i \rangle = \langle \sum_{i=1}^{n} \alpha_i Tv_i, \sum_{j=1}^{n} \alpha_j Tv_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha_j} \langle Tv_i, Tv_j \rangle$

$= \sum_{i=1}^{n} \alpha_i \bar{\alpha_i}$, as $Tv_i = w_i$ and $w_1 \dots w_n$ is an ON basis for V_2.

Thus $\langle v, v \rangle = \sum_{i=1}^{n} \alpha_i \bar{\alpha_i} = \langle Tv, Tv \rangle$. It follows $\|v\| = \|Tv\|$ for arbitrary v, and from (a), we know that this is equivalent to T being an isometry. Thus V_1 and V_2 are isometric.

In the other direction, let V_1 and V_2 be ~~vector~~ finite dim vector spaces such $\exists T$, an isometry, from V_1 to V_2. By definition, T is also an isomorphism. Then, we know from the Classification of Finite Dimensional Vector Spaces that $\dim V_1 = \dim V_2$.

7a. Recall that the k-th column of A is coordinate vector $[Tv_k]_C$, and similarly, the k-th column of B would be $[Sw_k]_B$.

If we let $Tv_i = \sum_{k=1}^{m} \alpha_k w_k$, then we have $\langle Tv_i, w_j \rangle = \langle \sum_{k=1}^{m} \alpha_k w_k, w_j \rangle$

$= \sum_{k=1}^{m} \alpha_k \langle w_k, w_j \rangle$. Because $w_1 \dots w_m$ are ON, this is equivalent to α_j.

Then, A_{ji} is the j-th element in the i-th column. From above fact, i-th column of A is $[Tv_i]_C$.

We let $Tv_i = \sum_{k=1}^{m} \alpha_k w_k$, so the i-th column is simply $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$ and the j-th element is α_j.

Similarly, Ae_i gives us the i-th column vector of A, and ~~A_{ij}~~ $Ae_i \cdot e_j$ extracts its j-th element. So, $\langle Tv_i, w_i \rangle = A_{ji} = Ae_i \cdot e_j = \alpha_j$.

Similarly, if we let $Sw_j = \sum_{k=1}^{n} \beta_k v_k$, then $\langle v_i, Sw_j \rangle = \langle v_i, \sum_{k=1}^{n} \beta_k v_k \rangle = \sum_{k=1}^{n} \overline{\beta_k} \langle v_i, v_k \rangle$.

Since v_i's form ON basis, we have $\langle v_i, Sw_j \rangle = \overline{\beta_i}$.

Note that B^*_{ji} is, by def'n, $\overline{B_{ij}} = \overline{(B_{ij})}$, the conjugate of the element in row i, column j of B. Column j of B is $[Sw_j]_B = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$, so the i-th element gives β_i, or $\overline{\beta_i}$ when conjugated.

Finally, $e_i \cdot Be_j = \overline{Be_j \cdot e_i}$, as $\cdot$ is an inner product. Be_j takes the j-th colum of B, and $Be_j \cdot e_i$ takes its i-th entry, so we have $\overline{Be_j \cdot e_i} = \overline{\beta_i}$. Thus all 3 are equivalent to $\overline{\beta_i}$.

b. Let T^* be the operator s.t. $[T^*]_{C,B} = A^*$. Then, we can show $\langle Tv_i, w_j \rangle = \langle v_i, T^* w_j \rangle$ by letting S be T^*. Then, as we defined it, $B = A^*$. Consider $B^*_{ji} = \overline{B_{ij}} = \overline{(A^*)_{ij}}$. Since $\overline{(\overline{z})} = z$, we have $\overline{A^*_{ij}} = \overline{\overline{A}}_{ji} = A_{ji}$. Then, from part (a), it follows

$$\langle Tv_i, w_j \rangle = A_{ji} = (B^*)_{ji} = \langle v_i, Sw_j \rangle = \langle v_i, T^* w_j \rangle,$$ as desired.

To show T^* is unique, we see that T^* was constructed on A^* so that $\begin{bmatrix} A_{1j} \\ \vdots \\ A_{nj} \end{bmatrix}$, the j-th column of A^*, satisfies $T^* w_j$. In other words, we defined T^* on the basis vectors of W, so it suffices to show that the ~~tra~~ linear transformation defined on the basis of a vector space is unique.

7b. (cont.)

Suppose T_1 and T_2 are both defined on $v_1 \dots v_n$, a basis of V. Then, $T_1 v_i = T_2 v_i$. We want to show $\forall v \in V$, $T_1 v = T_2 v$. Recall $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ for some $\alpha_1 \dots \alpha_n$, then

$$T_1 v = \alpha_1 T_1 v_1 + \dots + \alpha_n T_1 v_n = \alpha_1 T_2 v_1 + \dots + \alpha_n T_2 v_2 = T_2 v.$$ Thus T^* unique.

by linearity — by $T_1 v_i = T_2 v_i$ — by linearity

7c. Let v, w be arbitrary, so $v = \sum_{i=1}^{n} \alpha_i v_i$, and $w = \sum_{j=1}^{m} \beta_j w_j$. Then, $\langle Tv, w \rangle$

$$= \left\langle \sum_{i=1}^{n} \alpha_i T v_i, \sum_{j=1}^{m} \beta_j w_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \bar{\beta_j} \langle T v_i, w_j \rangle.$$ Recall from (b)

that $\langle T v_i, w_j \rangle = \langle v_i, T^* w_j \rangle$, so the above is equal to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \bar{\beta_j} \langle v_i, T^* w_j \rangle = \sum_{i=1}^{n} \alpha_i \left\langle v_i, T^*\left(\sum_{j=1}^{m} \beta_j w_j\right) \right\rangle$$

$$= \left\langle \sum_{i=1}^{n} \alpha_i v_i, T^*\left(\sum_{j=1}^{m} \beta_j w_j\right) \right\rangle = \langle v, T^* w \rangle,$$ by $\langle , \rangle$ linearity.

Thus $\langle Tv, w \rangle = \langle v, T^* w \rangle$, as desired.

7d. We wish to show the adjoint operator is unique. Suppose T^* and S^* both satisfy $\langle Tv, w \rangle = \langle v, T^* w \rangle$ and $\langle Tv, w \rangle = \langle v, S^* w \rangle$, for all $v \in V$ and $w \in W$.

In particular, this holds for $v = v_1, \dots, v_n$ and $w = w_1, \dots, w_m$, so we can say for all $i \in \{1 \dots n\}$ and $j \in \{1 \dots m\}$, $\langle T v_i, w_j \rangle = \langle v_i, T^* w_j \rangle$ and $\langle T v_i, w_j \rangle = \langle v_i, S^* w_j \rangle$. However, by part (b), T^* and S^* are unique, so $T^* = S^*$ necessarily. Thus T^* is unique.

7e. As it's defined, $T^{**}: V \to W$, and $\langle T^* w, v \rangle = \langle w, T^{**} v \rangle$. Also, $\langle Tv, w \rangle = \langle v, T^* w \rangle$.

Then, we conjugate the first one to see $\overline{\langle T^* w, v \rangle} = \overline{\langle w, T^{**} v \rangle}$, or $\langle v, T^* w \rangle = \langle T^{**} v, w \rangle$ by inner product properties. But, $\langle v, T^* w \rangle = \langle Tv, w \rangle$, so we have $\langle Tv, w \rangle = \langle T^{**} v, w \rangle$.

Subtract $\langle Tv, w \rangle$ from both sides to get: $\langle T^{**} v - Tv, w \rangle = 0$. This should hold for all $v \in V$ and $w \in W$, but in particular, it must hold for $w = T^{**} v - Tv$ (v can be arbitrary).

Then, we see $\langle T^{**} v - Tv, T^{**} v - Tv \rangle = 0 \Rightarrow T^{**} v - Tv = 0$ by vector space properties.

Thus for any v, $T^{**} v = Tv$, and so $T^{**} = T$, as desired.

8. Using the definition of invertibility from Hoffman & Kunze (1.6), it suffices to show $A^*A = I$ and $AA^* = I$. Recall $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ for matrix multiplication. Let A_i be the i-th column of A. Consider

$$(A^*A)_{ij} = \sum_{k=1}^{n} A^*_{ik} A_{kj} = \sum_{k=1}^{n} \bar{A}_{ki} \cdot A_{kj} = \sum_{k=1}^{n} A_{kj} \cdot \bar{A}_{ki}$$ by def'n of A^* and commutivity of multiplication. By inspection, $\sum_{k=1}^{n} A_{kj} \cdot \bar{A}_{ki} = A_j \cdot A_i$, which is 0 if $i \neq j$ as columns of A are orthogonal, and 1 if $i = j$, as columns are normal. Thus $(A^*A)_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases} \Rightarrow A^*A = I$,

as the only nonzero entries are 1 where $i=j$.

To show $AA^* = I$, we show A has rank n first. Note that $A^*A = I$, and I trivially has rank n, so A^*A also has rank n. Then, by Dimension Theorem, A^*A has kernel w/ dim 0, or only $A^*A(0) = 0$. Suppose, by contradiction, that A has rank less than n. Then, by Dimension Theorem, A has nonzero kernel: $\exists\, x \neq 0$ st $Ax = 0$. However, it follows that $A^*Ax = A^*(0) = 0$, so A^*A would also have nonzero kernel. This contradicts A^*A having kernel w/ dim 0, so A must have rank n.

Recall we showed $A^*A = I$, so $AA^*A = AI = A$. Then, subtracting A from both sides yields $AA^*A - A = 0$, or $(AA^* - I)A = 0$. Since A has rank n, this implies for all v, $(AA^* - I)v = 0$. Thus $AA^* - I$ is the zero matrix. $AA^* - I = 0 \Rightarrow AA^* = I$, as desired. Then, A^* is left and right inverse of A, so $A^{-1} = A^*$.

9a. Let $v_1, v_2 \in V$ be arbitrary. We wish to show $\langle T^*Tv_1, v_2\rangle = \langle v_1, T^*Tv_2\rangle$. Consider LHS, $\langle T^*Tv_1, v_2\rangle = \langle Tv_1, T^{**}v_2\rangle$ by def'n of T^{**}. But, in 7e, we showed $T^{**} = T \Rightarrow \langle Tv_1, Tv_2\rangle = \langle v_1, T^*Tv_2\rangle$ by def'n of T^*. Thus T^*T is hermitian, with $\langle T^*Tv_1, v_2\rangle = \langle v_1, T^*Tv_2\rangle$.

b. Note that by def'n of T^{**}, $\langle T^*Tv, v\rangle = \langle Tv, T^{**}v\rangle$. However, in 7e, we know $T^{**}v = Tv$. Thus $\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle$. By inner product properties, we know $\langle Tv, Tv\rangle \geq 0$, so $\langle T^*Tv, v\rangle \geq 0 \Rightarrow \langle T^*Tv, v\rangle$ is nonnegative.

10a. First, we note that if $v = w + w^{\perp}$, then $w = v_W = \sum_{i=1}^{n} \langle v, w_i \rangle w_i$ where $w_1 \dots w_n$ is an ON basis for W. So, $w^{\perp} = v - w \Rightarrow w - w^{\perp} = 2w - v$. Thus $Tv = 2w - v = 2v_W - v$.

First, we show T is Hermitian: $\langle Tv_1, v_2 \rangle = \langle v_1, Tv_2 \rangle$. Then, $\langle Tv_1, v_2 \rangle = \langle 2v_{1W} - v_1, v_2 \rangle$

$= \langle 2\sum_{i=1}^{n} (\langle v_1, w_i \rangle w_i) - v_1, v_2 \rangle = 2\sum_{i=1}^{n} \langle v_1, w_i \rangle \langle w_i, v_2 \rangle - \langle v_1, v_2 \rangle$ by linearity of inner products. Since $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and multiplication is commutative, this is equivalent to

$$2 \sum_{i=1}^{n} \overline{\langle v_2, w_i \rangle} \langle v_1, w_i \rangle - \langle v_1, v_2 \rangle$$

$$= 2 \sum_{i=1}^{n} \langle v_1, \langle v_2, w_i \rangle w_i \rangle - \langle v_1, v_2 \rangle \quad \text{because } \langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$$

$$= \langle v_1, 2\sum_{i=1}^{n} \langle v_2, w_i \rangle w_i \rangle - \langle v_1, v_2 \rangle = \langle v_1, 2\sum_{i=1}^{n} (\langle v_2, w_i \rangle w_i) - v_2 \rangle.$$

Note $\sum_{i=1}^{n} \langle v_2, w_i \rangle w_i = (v_2)_W$, so this is equivalent to $\langle v_1, 2(v_2)_W - v_2 \rangle = \langle v_1, Tv_2 \rangle$, as desired.

To show T is an isometry, we use the characterization $\|Tv\| = \|v\| \Leftrightarrow T$ is an isometry from Problem 6a. We can apply this because trivially $\dim V = \dim V$. Consider $\langle Tv, Tv \rangle = \langle 2v_W - v, 2v_W - v \rangle$. By linearity, this is equivalent to $4\langle v_W, v_W \rangle - 2\langle v_W, v \rangle - 2\langle v, v_W \rangle + \langle v, v \rangle$. We'll examine $\langle v_W, v_W \rangle$ first.

$$\langle v_W, v_W \rangle = \langle \sum_{i=1}^{n} \langle v, w_i \rangle w_i, \sum_{i=1}^{n} \langle v, w_i \rangle w_i \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v, w_i \rangle \overline{\langle v, w_j \rangle} \langle w_i, w_j \rangle.$$

Since $w_1 \dots w_n$ form an ON basis, this is equivalent to $\sum_{i=1}^{n} \langle v, w_i \rangle \overline{\langle v, w_i \rangle}$.

Then, we look at $\langle v_W, v \rangle = \langle \sum_{i=1}^{n} \langle v, w_i \rangle w_i, v \rangle = \sum_{i=1}^{n} \langle v, w_i \rangle \langle w_i, v \rangle = \sum_{i=1}^{n} \langle v, w_i \rangle \overline{\langle v, w_i \rangle}$.

Since $\langle v, v_W \rangle = \overline{\langle v_W, v \rangle}$, it follows $\langle v, v_W \rangle = \sum_{i=1}^{n} \overline{\langle v, w_i \rangle} \langle v, w_i \rangle$, or $\langle v, v_W \rangle = \langle v_W, v \rangle$.

In other words, $\langle v_W, v_W \rangle = \langle v, v_W \rangle = \langle v_W, v \rangle$, so it follows that

$$4\langle v_W, v_W \rangle - 2\langle v_W, v \rangle - 2\langle v, v_W \rangle = 0, \text{ so}$$

$\langle Tv, Tv \rangle = 0 + \langle v, v \rangle \Rightarrow \sqrt{\langle Tv, Tv \rangle} = \sqrt{\langle v, v \rangle}$, or $\|Tv\| = \|v\|$. Thus T is an isometry.

10b. We'll use the ~~two~~ following result ~~facts~~ from lecture:

Spectral Theorem: A hermitian operator $T: V \to V$, V is a fdips / $\mathbb{R}$ or $\mathbb{C}$, has an orthonormal basis of eigenvectors $\{v_1 \dots v_n\}$ for eigenvalues $\{\lambda_1 \dots \lambda_n\}$, eigenvalues not necessarily distinct. Furthermore, all eigenvalues are real.

Let $v_1 \dots v_n$ be a set of ON eigenvectors for T. We show that because T is also ~~is~~ an isometry, all $\lambda_i = \pm 1$. Note that by def'n of isometry, $\|Tv\| = \|v\|$ (showed in 6a). Consider ~~$v = v_1 \dots v_n$~~ $v = v_i$, $i \in \{1 \dots n\}$. Then, recall v_i is a normal vector, so $\|Tv\| = 1$. This also implies $\langle Tv_i, Tv_i \rangle = 1^2 = 1$. But, v_i is an eigenvector, so $Tv_i = \lambda v_i$, or $\langle \lambda_i v_i, \lambda_i v_i \rangle = 1$. Expand to get $\lambda_i \bar{\lambda}_i \langle v_i, v_i \rangle = 1$ or $\lambda_i \bar{\lambda}_i = 1$. Recall, though, that λ_i is real, so $\lambda_i^2 = 1 \Rightarrow \lambda_i = \pm 1$.

Let's reorder our basis, and denote a reordered ON basis as $w_1 \dots w_k, v_{k+1} \dots v_n$, where w_i is an associated eigenvector for $\lambda = 1$. $(0 \le k \le n)$. Let $W = \text{span}\{w_1 \dots w_k\}$. Recall in part (a) we showed that this $T(w + w^\perp) = w - w^\perp$ is equivalent to $Tv = 2v_W - v$. We show that W satisfies this.

Let v be arbitrary, but $\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=k+1}^{n} \alpha_j v_j = v$; which can be done because w_i's and v_j's are basis for V. Then,

$$Tv = T\left(\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=k+1}^{n} \alpha_j v_j\right) = \sum_{i=1}^{k} \alpha_i Tw_i + \sum_{j=k+1}^{n} \alpha_j Tv_j,$$

by linearity of T. Recall w_i's are all eigenvectors w/ $\lambda = 1$, and since v_j is also an eigenvector and it's eigenvalue is not 1, it must be ~~negative~~ -1. Making this substitution yields

$$Tv = \sum_{i=1}^{k} \alpha_i w_i - \sum_{j=k+1}^{n} \alpha_j v_j.$$

If we examine $2v_W - v$, we see that $v_W = \sum_{i=1}^{k} \langle v, w_i \rangle w_i$. Use the fact that $v = \sum_{i=1}^{k} \alpha_i w_i + \sum_{j=k+1}^{n} \alpha_j v_j$ to get

$$2v_W - v = 2\sum_{\ell=1}^{k} \left\langle \sum_{i=1}^{k} \alpha_i w_i + \sum_{j=k+1}^{n} \alpha_j v_j, w_\ell \right\rangle w_\ell - \left(\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=k+1}^{n} \alpha_j v_j\right)$$

$$= 2\sum_{\ell=1}^{k} \alpha_\ell \langle w_\ell, w_\ell \rangle w_\ell - \left(\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=k+1}^{n} \alpha_j v_j\right)$$

after expanding with linearity of inner product and using the fact that $w_1 \dots w_k, v_{k+1} \dots v_n$ are ~~linearly~~ orthonormal.

$$\Rightarrow 2v_W - v = 2\sum_{\ell=1}^{k} \alpha_\ell w_\ell - \sum_{i=1}^{k} \alpha_i w_i - \sum_{j=k+1}^{n} \alpha_j v_j = \sum_{\ell=1}^{k} \alpha_\ell w_\ell - \sum_{j=k+1}^{n} \alpha_j v_j,$$

which is exactly the same as Tv. Thus $Tv = 2v_W - v$, ~~and T is the reflection over W~~, where W is the subspace spanned by all eigenvectors of T w/ $\lambda = 1$.