Math 115AH Final

Pochao Yang

204831541

1. State and prove the Replacement Theorem.

2. Let V be a finite dimensional vector space over a field F and $\mathcal{B}$ a subset of V. Prove the following are equivalent:
   (a) $\mathcal{B}$ is a basis for V. $a \to b$ ✓
   (b) $\mathcal{B}$ is a minimal spanning set for V. $c \to a$ ✓
   (c) $\mathcal{B}$ is a maximal linearly independent set for V. $b \to c$ ✓

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a counterclockwise rotation by an angle θ in the plane perpendicular to $(1,1,2)$ and $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a counterclockwise rotation by an angle φ in the plane perpendicular to $(-1,0,1)$. Find $[R \circ T]_{\mathcal{S}}$ where $\mathcal{S}$ is the standard basis for $\mathbb{R}^3$.
   [You need not multiply out matrices but must compute the necessary inverses.]

4. Let V be a vector spaces over a field F and W_1 and W_2 finite dimensional subspaces. Recall if $S \subset V$, then $S^0 := \{f \in V^* \mid f(s) = 0 \text{ for all } s \text{ in } S\}$, the *annihilator* of S. Prove both of the following:
   (a) $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.
   (b) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

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5. Let V be an n-dimensional vector space over the rational numbers $\mathbb{Q}$ and $T : V \to V$ a linear operator. Suppose that T is *nilpotent*, i.e., for every v in V there exists a positive integer N such that $T^N v = 0$. Prove the following:
   (a) If there exists a vector v such that $T^m v = 0$ but $T^{m-1} v \neq 0$, then $\{v, Tv, \ldots, T^{m-1}v\}$ is linearly independent.
   (b) T is triangularizable. discussion

6. Let V be a vector space over a field F, $W_i \subset V$ subspaces of V, $i \in I$, and $W = \sum_{i \in I} W_i$. Define what it means for W to be the *direct sum* of the W_i, $i \in I$. Suppose $T : V \to V$ is a linear transformation and the W_i are distinct eigenspaces of T. Using only the appropriate definitions, (i.e., no theorems, etc.), prove that W is a direct sum of the W_i. (Note there is no finiteness assumptions about anything. If you use any finiteness assumptions, you cannot get full credit.)

7. State and prove the (full) Orthogonal Decomposition Theorem.

8. Let V be the real inner product space of continuous functions with inner product given by $\langle f, g \rangle = \int_0^{2\pi} fg$. Compute the explicit Fourier approximation of $f(x) = x + 1$ onto $\mathrm{Span}(\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(x))$, the usual first three orthonormal Fourier functions. What is the error? You do not have to explicitly compute the error.

9. Let V be a finite dimensional real inner product space and $T : V \to V$ a hermitian operator. Prove that there exists a unique linear operator $S : V \to V$ such that $T = S^3$. $S = T^t$

10. Prove Schur's Theorem which says: Let V be a finite dimensional complex vector space and $T : V \to V$ a linear operator, then there exists an orthonormal basis for V such that $[T]_{\mathcal{B}}$ is upper triangular.

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11. Let V be a finite dimensional inner product space over a field F and $T : V \to V$ a linear operator. Assume that the adjoint $T^* : V \to V$ exists. Show that T has an eigenvector if and only if T^* has an eigenvector.

λ is a root of f_T

$\ker T \neq \{0\} \Rightarrow \ker T^* \neq \{0\}$

$\bar{\lambda}$ is a root of f_{T^*}

$\ker(A - \lambda I) \neq \{0\}$

$\langle Tv, v \rangle = \langle \lambda v, v \rangle$

$= \langle v, \bar{\lambda} v \rangle$