Math 115AH
Midterm I
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4. 6

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Extra Credit Problem: ~~10~~ 9

You can use the theorems from the book or proved in class, but you should indicate which theorems you are using. You can use the fact that a vector space with a finite spanning set has a basis. **If you give a counterexample in the proof or counterexample problems, you should explain why your example is a counterexample**

V and W are finite dimensional vector spaces over a field F. Both V and W are not the zero vector space. You can assume F is a subfield of C.

If $T : V \to W$ is linear, $R(T)$ is the range of T and $N(T)$ is the null space of T.

1) a) Suppose $\alpha_1, \alpha_2, \dots \alpha_n \in V$. Define what it means to say $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ span V. Be very careful to use complete sentences and phrases such as "there exist" (or $\exists$) and "for all" (or $\forall$). F is the scalars.

$\alpha_1, \dots, \alpha_n$ span V if, for all $v \in V$, there exist a set of scalars $a_1, \dots a_n \in F$ (where F is the field of V) such that

$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n = v$.

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b) Define what it means to say $\alpha_1, \alpha_2, \dots \alpha_n$ are linearly independent. Complete sentences!

$\alpha_1, \dots \alpha_n$ are linearly independent if, for all $a_1, \dots, a_n$ in F the field of V,

$a_1 \alpha_1 + \dots + a_n \alpha_n = 0$ implies $a_1 = a_2 = \dots = a_n = 0$.

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2) Suppose α, β, γ are linearly independent. Show $\alpha+\beta$, $\beta+\gamma$ and $\gamma+\alpha$ are linearly independent. Say what you are doing in complete sentences, don't just compute.

We must show that for all c_1, c_2, c_3 where

$c_1(\alpha+\beta) + c_2(\beta+\gamma) + c_3(\gamma+\alpha) = 0$,

$c_1 = c_2 = c_3 = 0$. We can rewrite this equation

as $\alpha(c_1+c_3) + \beta(c_2+c_1) + \gamma(c_2+c_3) = 0$.

Since α, β, γ are linearly independent,

$c_1 + c_3 = c_2 + c_1 = c_2 + c_3 = 0$.

$c_1 + c_3 = c_2 + c_1$ implies $c_2 = c_3$, and

$c_2 + c_3 = c_1 + c_3$ implies $c_1 = c_2$.

Furthermore, $c_1 + c_3 = 0$ implies $c_1 = -c_3$,

so $c_2 + c_3 = c_2 + c_3$ which implies

$2c_2 = 0$. Hence $c_2 = 0$, and $c_1 = 0$ and

$c_3 = 0$.

Thus $\alpha+\beta$, $\beta+\gamma$, and $\gamma+\alpha$ are linearly independent.

S ~~is~~ a basis of W

$S(v_1) \quad \dots \quad S(w_n)$

3) Suppose V and W are finite dimensional and $T : V \to W$ is linear and surjective. Show there is a linear $S : W \to V$ so that TS is the identity on W:

$$TS(\beta) = \beta$$

for $\beta \in W$. Hint: Define S on a basis of W.

Let $\beta_1 := \{v_1, \ldots, v_n\}$ be a basis of V.

Then since T ~~is~~ surjective, $\beta_2 := \{T(v_1), \ldots T(v_n)\}$ is a basis ~~for~~ W. Hence $\dim V = \dim W$.

Therefore, ~~there~~ exists an S that can map a basis for W to a basis for V, i.e. for each vector v in β_1, $S(w) = v$ where $w \in \beta_2$. Since β_2 is a basis, S is surjective.

Then $TS(\beta) = \beta$ for all $\beta \in W$.

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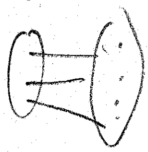

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4a) Suppose $T : V \to V$ and $S : V \to V$ are linear and V is finite dimensional. Suppose TS is injective (one to one). Show T is injective.

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Suppose $\beta = S(\alpha)$.

Then suppose $T(\beta) = 0$. Since $\beta = S(\alpha)$, we have that $TS(\alpha) = 0$. Since TS is injective, $\ker(TS) = 0$, so $\alpha = 0$. Hence, $S(v) = \beta = 0$.

Therefore $T(\beta) = 0 \Rightarrow \beta = 0$, i.e. T is injective.

Why can β be of the form $S(\alpha)$?

4b) Give an example of a vector space V and a linear transformations $T : V \to V$ and $S : V \to V$ so that TS is injective, but T is not injective.

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Let $V = \mathbb{N}$ over $\mathbb{R}$. ? $\mathbb{N}$ is not a vector space ...

Let $S : V \to V$ be $T(v) = 2v$.

Let $T : V \to V$ be $S(v) = \lfloor \frac{1}{2} v \rfloor$

(where $\lfloor \ \rfloor$ is floor function)

Then $TS(v) = v$ for all $v \in \mathbb{N}$,

but T is not injective, since $T(3) = T(2) = 1$.

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5) Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ are six distinct elements of V. Suppose $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of a subspace $W_1 \subset V$ and $\{\alpha_4, \alpha_5, \alpha_6\}$ is a basis of a subspace $W_2 \subset V$. Suppose $W_1 \cap W_2 \neq \{0\}$. True or false: $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ is linearly dependent. Proof or counterexample.

This is true. Pf:

These are fixed!

this does not make sense

we must show that for all $\alpha_1, \ldots, \alpha_6$, if $c_1\alpha_1 + \cdots + c_6\alpha_6 = 0$, then there exist $c_1, \ldots, c_6$ in F where at least one c_i is nonzero

Let v be a non zero vector in $W_1 \cap W_2$. Then since $v \in W_1$, $v = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$ for some a_1, a_2, a_3. Since $v \in W_2$, $v = a_4\alpha_4 + a_5\alpha_5 + a_6\alpha_6$ for some a_4, a_5, a_6. Since v is non zero, we have that at least one from a_1, a_2, a_3 is non zero.

Now we can write $v = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = a_4\alpha_4 + a_5\alpha_5 + a_6\alpha_6$. This yields $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 - a_4\alpha_4 - a_5\alpha_5 - a_6\alpha_6 = 0$. Since at least one of a_1, a_2, a_3 is non zero, we are done. ■ ✓

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Extra credit (10pts): Suppose $T : V \to V$ is linear and V is finite dimensional. Suppose $T^2 = T$. Show $V = R(T) + N(T)$ and $R(T) \cap N(T) = \{0\}$.

1) Suppose $v \in R(T) \cap N(T)$. Then $T(v) = 0$ and $T(w) = v$. Then $T(T(w)) = T(v) = T(w)$.

who is w ?

Hence $T(w) = 0$, or $v = 0$,

So $R(T) \cap N(T) = 0$. ✓

2)

By the rank-nullity theorem, $\dim(R(T)) + \dim((N(T)) = \dim(V)$

$R(T) + N(T) \subseteq V$.

By Grassmann's Formula

$\dim((R(T)) + \dim(N(T)) = \dim(R(T) + N(T)) \cdot \dim(R(T) \cap N(T))$

which implies $\dim(R(T) + N(T)) = \dim(V)$.

since $R(T) + N(T) \subseteq V$, $R(T) + N(T) = V$.

✓

Scratch