1) a) Suppose $\alpha_1, \alpha_2, \ldots \alpha_n \in V$. Define what it means to say $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ span V. Be very careful to use complete sentences and phrases such as "there exist" (or $\exists$) and "for all" (or $\forall$). F is the scalars.

For any $v \in V$, $\exists$ $a_1, a_2 \ldots a_n \in F$ such that $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = v$.

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b) Define what it means to say $\alpha_1, \alpha_2, \ldots \alpha_n$ are linearly independent. Complete sentences!

The only $c_1 \ldots c_n \in F$ that satisfy the equation $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0$ are $c_1 = 0$, $c_2 = 0$, $c_3 = 0 \ldots c_n = 0$

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2) Suppose α, β, γ are linearly independent. Show $\alpha+\beta$, $\beta+\gamma$ and $\gamma+\alpha$ are linearly independent. Say what you are doing in complete sentences,don't just compute.

Let $c_1, c_2, c_3 \in F$.

$$c_1(\alpha+\beta) + c_2(\beta+\gamma) + c_3(\gamma+\alpha) = 0$$

$$c_1\alpha + c_1\beta + c_2\beta + c_2\gamma + c_3\gamma + c_3\alpha = 0$$

Rearranging and using the distributive property we get

$$(c_1+c_3)\alpha + (c_1+c_2)\beta + (c_2+c_3)\gamma = 0$$

However, since α, β, γ are linearly independent, the equation is satisfied only when

$$c_1+c_3=0, \quad c_1+c_2=0 \quad c_2+c_3=0$$

So $c_1=-c_3$ from the first eq'n and $c_2=-c_3$ from third eq'n. By substituting into second eq'n.

$$-c_3 - c_3 = 0$$

$$-2c_3 = 0$$

$$c_3 = 0$$

So $c_1=-0=0$ and $c_2=-0=0$. So the only way to satisfy original eq'n is if $c_1=c_2=c_3=0$. Thus the set $\{\alpha+\beta, \beta+\gamma, \gamma+\alpha\}$ are linearly independent.

injective

TS

$R(T) = W$

3) Suppose V and W are finite dimensional and $T : V \to W$ is linear and surjective. Show there is a linear $S : W \to V$ so that TS is the identity on W:

$$TS(\beta) = \beta$$

for $\beta \in W$. Hint: Define S on a basis of W.

Since T is surjective, we claim $\dim V \geq \dim W$.

$TS: V \to V$

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4a) Suppose $T : V \to V$ and $S : V \to V$ are linear and V is finite dimensional. Suppose TS is injective (one to one). Show T is injective.

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$TS: V \to V$

$N(TS) = \{0\}$ since it is injective

$\text{nullity}(TS) + \text{rank}(TS) = \dim V$

$\text{rank}(TS) = \dim V$

$R(TS) = V$

For any element $v \in V$, $\exists\ \alpha \in V$ such that $T(S(\alpha)) = x$

Thus $\forall\ v \in V$, $\exists\ \beta = S(\alpha)$ such that $T(\beta) = x$.

Thus T is onto. $R(T) = V$. Rank $T = V$. By rank nullity

$\text{rank}\,T + \text{nullity}\,T = \dim V$

$\dim V + \text{nullity}\,T = \dim V$

$\text{nullity}\,T = 0$. Thus $N(T) = \{0\}$ and T is injective

4b) Give an example of a vector space V and a linear transformations $T : V \to V$ and $S : V \to V$ so that TS is injective, but T is not injective.

$V =$ vector space of all polynomials.

V has an infinite basis $\{1, x^1, x^2, x^3 \ldots\}$

So it is infinite dimensional.

$T(\alpha) =$

lo

5) Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ are six distinct elements of V. Suppose $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of a subspace $W_1 \subset V$ and $\{\alpha_4, \alpha_5, \alpha_6\}$ is a basis of a subspace $W_2 \subset V$. Suppose $W_1 \cap W_2 \neq \{0\}$. True or false: $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ is linearly dependent. Proof or counterexample.

False. let $V = \mathbb{R}^6$

let $W_1 = (a, b, c, 0, 0, 0)$

let $W_2 = (0, 0, 0, d, e, f)$

$\{e_1, e_2, e_3\}$ are a basis of W_1

$\{e_4, e_5, e_6\}$ are a basis of W_2

the set $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is linearly independent since $e_1 \ldots e_6$ ~~are~~ is the standard basis and is trivially linearly independent

but $W_1 \cap W_2 = \{0\}$ in your example..

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Extra credit (10pts): Suppose $T : V \to V$ is linear and V is finite dimensional. Suppose $T^2 = T$. Show $V = R(T) + N(T)$ and $R(T) \cap N(T) = \{0\}$.

Since $T^2 = T$, $N(T^2) = N(T)$

Let $x \in R(T) \cap N(T)$.

By the def $x = T(\alpha)$ for some $\alpha \in V$ and $T(x) = 0$. $T(T(\alpha)) = 0$. So $\alpha \in N(T)$ $\alpha \in N(T^2)$. So $x = T(T(\alpha)) = 0$ is the only element. So $R(T) \cap N(T) \subseteq \{0\}$.

Since $R(T) \cap N(T) \supseteq \{0\}$ (since both are subspaces), $R(T) \cap N(T) = \{0\}$.

Let $v \in V$. We know trivially that $R(T) + N(T) \subseteq V$.