Math 115AH
Midterm I
November 3, 2017

Pleas

Signature:

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Extra Credit Problem: 10

You can use the theorems from the book or proved in class, but you should indicate which theorems you are using. You can use the fact that a vector space with a finite spanning set has a basis. **If you give a counterexample in the proof or counterexample problems, you should explain why your example is a counterexample**
V and W are finite dimensional vector spaces over a field F. Both V and W are not the zero vector space. You can assume F is a subfield of C.

If $T: V \to W$ is linear, $R(T)$ is the range of T and $N(T)$ is the null space of T.

1) a) Suppose $T : V \to V$ is linear. Define what it means for T to be onto, a.k.a sujective. Be very careful to use complete sentences and phrases such as "there exist" (or $\exists$) and "for all" (or $\forall$).

If T is onto,

for all $\alpha \in V$, $\exists\, \beta \in V$ such that $T\beta = \alpha$.

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$R(T) = V$

b) Suppose $\alpha_1, \alpha_2, \ldots \alpha_n \in V$. Define what it means to say $\alpha_1, \alpha_2, \ldots \alpha_n$ are linearly independent. Complete sentences!

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent,

the equation with scalars $c_1, \ldots, c_n$

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$$

is satisfied only if $c_1 = c_2 = \cdots = c_n = 0$.

(trivial solution)

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2) Suppose the dimension of V is n. Suppose $T : V \to W$ is linear and suppose $\{\alpha_1, \alpha_2, \dots \alpha_k\} \subset V$. Either provide a tight rigorous proof with complete sentences if the following are true or an explicit counterexample.

a) True or false: If $T(\alpha_1), T(\alpha_2), \dots T(\alpha_k)$ are linearly independent, then T is one to one, aka injective.

~~True. We show if $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_k)$ are linearly independent, then $T(\beta_1) = T(\beta_2) \Rightarrow \beta_1, \beta_2$ for all $\beta_1, \beta_2 \in V$.~~

False. Let $k=2$.

Let $\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$; $\alpha \mapsto \alpha^2$, **not linear.**

It is obvious that $T\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T\alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent.

However, $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

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(We also see that T is not non-singular i.e. $N(T) \neq \{0\}$ so it is not one-to-one).

$\Big(1 \; 0$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

b) True or false: If $T(\alpha_1), T(\alpha_2), \dots T(\alpha_k)$ span W, then T is onto.

True. We show if $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_k)$ span W, T is onto (i.e. for all $w \in W$, $\exists\, \alpha \in V$ such that $T\alpha = w$) i.e. range of $T = W$.

$\beta = 0$

Since $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_k)$ span W, for all $w \in W$,

$w = c_1 T(\alpha_1) + c_2 T(\alpha_2) + \dots + c_k T(\alpha_k)$ for some scalars $c_1, c_2, \dots, c_k \in \mathbb{C}$.

Since T is linear,

$w = T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_k\alpha_k)$

Since V is a vector space, and $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subset V$,

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_k\alpha_k \in V$.

So $T\alpha = w$ and for all $w \in W$, $\exists\, \alpha \in V$ s.t. $T\alpha = w$ (i.e. T is onto).

~~By definition, T~~

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3) Suppose $\dim V = 2$ and $\dim W = 3$ and that $T : V \to W$ and $U : W \to V$ are linear.

a) Can you find an example of U and T so that UT is invertible? Either find an example or prove that there is no such example.

$T: \mathbb{R}^2 \to \mathbb{R}^3 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$U: \mathbb{R}^3 \to \mathbb{R}^2 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$

$UT: \mathbb{R}^2 \to \mathbb{R}^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ equivalent to Id.

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b) Can you find an example of U and T so that TU is invertible? Either find an example or prove that there is no such example.

No, TU cannot be invertible.

It suffices to show TU cannot be onto.

Assume by contradiction that TU is onto.

Then, T is also onto, by a theorem for linear transformations T, U.

Let $\{\alpha_1, \alpha_2\}$ be a basis for V.

(A theorem states that any bases a vector space of finite dimension n have exactly n vectors.)

Then, $\{T\alpha_1, T\alpha_2\}$ is a basis for W (which by a theorem, T is onto is equivalent to the statement "if $\{\alpha_1, \alpha_2\}$ be a basis for V, $\{T\alpha_1, T\alpha_2\}$ is basis for W for linear $T: V \to W$")

But $2 < \dim W = 3$; $\{T\alpha_1, T\alpha_2\}$ cannot be a basis for W.

so we have a contradiction

Thus, T cannot be onto, TU cannot be onto.

Hence, TU cannot be invertible. $\square$

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$f \circ g$ onto
f onto

$f \circ g$ 1-to-1
g 1-to-1

4) Suppose $T : V \to V$ is linear and V has dimension n.

$R(\cdot) \to$

a) Suppose $n = 3$. Can you find an example of a T so that $R(T) = N(T)$? Either find an example or show that none exists.

None exists.

By the rank-nullity theorem, $\dim R(T) + \dim N(T) = \dim V$

Assume by contradiction $R(T) = N(T)$.

Then, $\dim R(T) = \dim N(T)$.

Then, by the rank-nullity theorem $\dim R(T) + \dim N(T) = \dim R(T) + \dim R(T) = \dim V = 3$

But, then $2 \dim R(T) = 3$ and $\dim R(T)$ must be an integer, so we have a contradiction.

Thus, no such T exists. $\square$

b) Suppose $n = 2$. Can you find an example of a T so that $R(T) = N(T)$? $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to$ Either find an example or show that none exists.

The rank-nullity theorem tells us that $\dim R(T) + \dim N(T) = \dim V = 2$

Thus, rank(T) = nullity$(T) = 1$, assuming there exists such a T as requested.

Let $\left\{ T : \mathbb{R}^2 \to \mathbb{R}^2 \; ; \; x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \right\}$

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $R(T)$ (e.g. $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives us $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$)

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is ~~as~~ a basis for $N(T)$

$R(T) = N(T)$.

5) Suppose W_1 and W_2 are two subspaces of V and that α_1 and α_2 are a basis of W_1 and that β_1 and β_2 are a basis of W_2. Suppose $W_1 \cap W_2 = \{0\}$. True or false: The set $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ is linearly independent. Proof or counterexample.

True

We show that $W_1 \cap W_2 = \{0\} \Rightarrow \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ are linearly independent.

Since α_1, α_2 are basis of W_1, $c_1\alpha_1 + c_2\alpha_2 = 0$ iff $c_1 = c_2 = 0$.

Since β_1, β_2 are basis of W_2, $d_1\beta_1 + d_2\beta_2 = 0$ iff $d_1 = d_2 = 0$.

Suppose $\alpha \in W_1 \cap W_2$. Then, $\alpha \in W_1$ and $\alpha \in W_2$.

Then $\alpha = c_1\alpha_1 + c_2\alpha_2 = d_1\beta_1 + d_2\beta_2$ for some scalars $c_1, c_2, d_1, d_2 \in \mathbb{C}$.

Since we are given $W_1 \cap W_2 = \{0\}$, $\alpha = 0$, so

$c_1\alpha_1 + c_2\alpha_2 = d_1\beta_1 + d_2\beta_2$

$c_1\alpha_1 + c_2\alpha_2 = d_1\beta_1 + d_2\beta_2 = 0$

Prob: Only shows $c_1 = c_2 = d_1 = d_2 = 0$ if $c_1\alpha_1 + c_2\alpha_2 = d_1\beta_1 + d_2\beta_2$

This is only satisfied if $c_1 = c_2 = d_1 = d_2 = 0$ (from above).

We want to show $c_1\alpha_1 + c_2\alpha_2 + d_1\beta_1 + d_2\beta_2 = 0$ iff $c_1 = c_2 = d_1 = d_2 = 0$

Rearranging the equation, we have that

$c_1\alpha_1 + c_2\alpha_2 - d_1\beta_1 - d_2\beta_2 = 0$ is only satisfied if $c_1 = c_2 = d_1 = d_2 = 0$.

Rearranging the equation, we have

$c_1\alpha_1 + c_2\alpha_2 = -d_1\beta_1 - d_2\beta_2 = -(d_1\beta_1 + d_2\beta_2)$

Hence, $\{c_1, c_2, d_1, d_2\}$ are linearly independent.

$\alpha_1, \alpha_2, \beta_1, \beta_2$

But $c_1\alpha_1 + c_2\alpha_2 = d_1\beta_1 + d_2\beta_2$ iff $c_1 = c_2 = d_1 = d_2 = 0$

$c_1\alpha_1 + c_2\alpha_2 + c_3\beta_3 + c_4\beta_2 = 0$

as above.

Alt. approach: Show $W_1 + W_2$ is subspace w/ dim 4, $\alpha_1, \alpha_2, \beta_1, \beta_2$ span $W_1 + W_2$, so it is basis $\Rightarrow$ so linearly ind.

Extra credit (10pts): Suppose $\dim V = 4$ and that $T : V \to V$ is linear. Suppose the rank of T is two, i.e. the dimension of $\dim R(T) = 2$. Suppose $T^2 = T$. Show there is a basis $\mathcal{B}$ of V so that $[T]_{\mathcal{B}}$ is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad [T]_{\mathcal{B}} = \left([T(1)]_{\mathcal{B}}, [T_2]_{\mathcal{B}} \right)$$

We show if $T^2 = T$, $\exists$ basis $\mathcal{B}$ of V s.t. $[T]_{\mathcal{B}} = A$.

By the rank-nullity theorem, $\dim R(T) + \dim N(T) = \dim V$

and thus $\dim N(T) = \dim V - \dim R(T) = 4 - 2 = 2$.

Let $\{\alpha_1, \alpha_2\}$ be a basis for $R(T)$ and $\{\beta_1, \beta_2\}$ be a basis for $N(T)$.

We show for all $\alpha \in R(T)$, $T\alpha = \alpha$.
Let $\alpha \in R(T)$.
Then, $T\gamma = \alpha$ for some $\gamma \in V$.
Since $T^2 = T$,
$T\alpha = T(T\gamma) = T^2\gamma = T\gamma = \alpha$.
Thus, for all $\alpha \in R(T)$, $T\alpha = \alpha$.

Then, $T\beta_1 = T\beta_2 = 0$.
Since $\alpha_1, \alpha_2 \in R(T)$, $T\alpha_1 = \alpha_1$, $T\alpha_2 = \alpha_2$.

It is obvious that $\mathcal{B} = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ is an ordered basis of V. (we simply show they are linearly independent using $T^2 = T$).

We have $T\alpha_1 = \alpha_1$
$T\alpha_2 = \alpha_2$
$T\beta_1 = 0$
$T\beta_2 = 0$

Thus,

$$[T]_{\mathcal{B}} = \left[[T(\alpha_1)]_{\mathcal{B}} \; [T(\alpha_2)]_{\mathcal{B}} \cdots [T\beta_2]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$T\alpha_1 = \alpha_1$
$T\gamma_1$