Math 115AH
Final
December 10, 2010

Name:____________________, ____________________
Please put your last name first and print clearly

Signature:______________________________

1.__________	2. __________
3.__________	4. __________
5.__________	6. __________
7.__________	8. __________
9.__________	10. __________
11. __________	12. __________

Please do not use the spaces on this page.

NO CALCULATORS!

V is a vector space over a field F in the following problems. You can assume F is the reals or the complex numbers. If $T : V \to W$ is linear, then

$$R(T) = \{w \in W | w = T(v) \text{ for some } v \in V\}$$

and

$$N(T) = \{v \in V | 0 = T(v)\}$$

1. If S is a subset of V, we define S^0 is the set of all elements of V^* which vanish on S.

a) If $V = \mathbf{R}^3$ and

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Find a basis of S^0

b) If W_1 and W_2 are subspaces of V of dimension n, find a relation between W_1^0, W_2^0 and $(W_1 + W_2)^0$. Prove your answer. Best credit for most precise relation.

2. Suppose $T : V \to W$ and $U : W \to V$ and $0 < \dim V < \dim W$. Assume that T and U are linear transformations.

a) Can UT be an isomorphism (one to one and onto)? Prove your answer.

b) Can TU be an isomorphism? Prove your answer.

3. In this problem, the field is $\mathbf{C}$. Suppose V has an inner product $\langle x, y\rangle$. $T : V \to V$ is a linear transformation. T is said to be anti-selfadjoint if

$$\langle x, Ty\rangle = -\langle Tx, y\rangle$$

a)If $\mathcal{B}$ is an orthonormal basis of V, what can you say about the entries of matrix $[T]_{\mathcal{B}}$. Prove your answer.

b) If W is T invariant, what can you say about $W^{\perp}$? Prove your answer.

4. With reference the definition in Problem 3, show that any normal linear transformation T is sum of a self adjoint linear transformation T_1 and anti-self adjoint linear transformation T_2

$$T = T_1 + T_2$$

so that T_1 and T_2 commute:

$$T_1 T_2 = T_2 T_1.$$

Hint: $T + T^*$ is self adjoint.

5. If $v \in \mathbf{R}^n$ is a eigenvector of a matrix A and P is a invertible matrix, how would you find an eigenvector w of PAP^{-1}?

6. If V has dimension 3 and $T^3 = 0$, but $T^2 \neq 0$, show there is a basis $\mathcal{B}$ so that $[T]_{\mathcal{B}}$ has entries 0 or 1. Be sure to prove you basis really is a basis. Determine $[T]_{\mathcal{B}}$ explicitly.

7.In this problem, $F = \mathbf{C}$. Give an example of a 2×2 matrix which cannot be diagonalized . Prove your matrix cannot be diagonalized.

8. Let S and T be linear transformations from finite dimensional V to finite dimensional W. Find a inequality relation between $\dim R(T)$, $\dim R(S)$ and $\dim R(T+S)$. Prove your answer. Best credit for the most precise answer.

9. In this problem, $F = \mathbf{C}$. Suppose that $T : V \to V$ is unitary

$$\langle x, y \rangle = \langle Tx, Ty \rangle$$

Suppose λ is an eigenvalue of T.

a) What can you say about $|\lambda|$, the absolute value of λ.

b) If T is both unitary and self adjoint, what is T^2?

10. Let V be a vector space of dimension n and $T : V \to V$ be a linear transformation. Let W_k be the image (range) of T^k. Let S_k be the kernel of T^k.

a) True or false: All the S_k are T invariant. Prove or provide a counterexample.

b) True or false: All the W_k are T invariant. Prove or provide a counterexample.

11) Continuing 10: Let V be a vector space of dimension n and $T : V \to V$ be a linear transformation. Let W_k be the image (range) of T^k. Let S_k be the kernel of T^k.
a) Prove there is a $K \in \mathbf{Z}$ so that

$$S_K = S_{K+1} = S_{K+2} \dots$$

Hint: Study $\dim S_k$ and inclusion relations among the S_k

b) What can you say about $S_K \cap W_K$ and $S_K + W_K$? Prove your answer.

12. Continuing 11: Prove any $n \times n$ matrix can be written as in block form:

$$\begin{pmatrix} N & 0 \\ 0 & B \end{pmatrix}$$

where N is a $k \times k$ nilpotent matrix ($N^n = 0$) and B is an $(n-k) \times (n-k)$ invertible matrix.