Math 115A
Linear Algebra

Midterm

Instructions: You have ~~40~~ 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly and justify your answers. Please write your solutions in the space below the questions; you can go over the page and continue on the back; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name and UID in the space below.

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Number of additional sheets attached: __________

Question	Points	Score
1	10	10
2	10	10
3	10	10
4	10	10
Total:	40	40

Problem 1. *10pts.*

Prove by induction that for every natural number $n \geq 1$

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = 2\left(1 + (n-1)2^n\right).$$

Base case: $n=1$ $\quad n \cdot 2^n = 1 \cdot 2^1 = 2$

$$2(1+(n-1)2^n) = 2(1+(1-1)2^1) = 2$$

Inductive step: Assume claim is true for k. ($k \geq$ base case #)

$$1\cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + k \cdot 2^k = 2(1+(k-1)2^k) \text{ true.}$$

Want to prove claim for $k+1$.

$$1\cdot 2 + 2\cdot 2^2 + 3 \cdot 2^3 + \ldots + (k+1)\cdot 2^{k+1} \quad (\ldots = 2(1+(k+1-1)2^{k+1}))$$

$$= 1\cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + k \cdot 2^k + (k+1)\cdot 2^{k+1}$$

$$= 2(1+(k-1)2^k) + (k+1)\cdot 2^{k+1}$$

$$= 2(1+(k-1)2^k + (k+1)\cdot 2^k)$$

$$= 2(1+(k-1+k+1)\cdot 2^k)$$

$$= 2(1+(2k)\cdot 2^k)$$

$$= 2(1+k\cdot 2^{k+1})$$

$$= 2(1+(k+1-1)2^{k+1})$$

Problem 2.

Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space over the field $\mathbb{R}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) [7pts.] Prove that the following subset is a subspace of V:

$$W = \{f \in V \mid f(\pi) = f(e) = 0\}.$$

(b) [3pts.] Prove that the following is *not* a subspace of V:

$$Z = \{f \in V \mid f(\pi) = f(e) = 1\}.$$

(a) $\vec{0}$ is $h(t) = 0$

$h(\pi) = h(e) = 0$

So $\vec{0} \in W$

Let $x, y \in W$ arbitrary.

$(x+y)(t) = x(t) + y(t) \quad \forall t \in \mathbb{R}$

$(x+y)(\pi) = x(\pi) + y(\pi)$
$= 0 + 0$
$= 0$

$(x+y)(e) = x(e) + y(e)$
$= 0 + 0$
$= 0$

So, $(x+y) \in W$

Let $x \in W, c \in \mathbb{R}$ arbitrary.

$(cx)(t) = cx(t) \quad \forall t \in \mathbb{R}$

$(cx)(\pi) = cx(\pi)$
$= c0$
$= 0$

$(cx)(e) = cx(e)$
$= c0$
$= 0$

So, $(cx) \in W$

$\therefore$ W is a subspace of V. $\square$

(b) To be a subspace,

$\forall x, y \in Z,\ x+y \in Z$.

Let $x, y \in W$ be arbitrary.

$(x+y)(t) = x(t) + y(t) \quad \forall t \in \mathbb{R}$

$(x+y)(\pi) = x(\pi) + y(\pi)$
$= 1 + 1$
$= 2 \neq 1$

So, $(x+y) \notin Z$

$\Rightarrow$ Z not a subspace. $\square$

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Problem 3.

Let $V = P_3(\mathbb{Q})$ be the vector space of polynomials with $\mathbb{Q}$ coefficients of degree at most 3. Let $S = \{x+1, x^3+x, 2x^3+1\}$.

(a) [5pts.] Is S a linearly dependent set or not? Justify your answer.

(b) [5pts.] For each of the following polynomials, determine whether they belong to span S. Justify all your answers.

(i) $3x^3+3$;

(ii) x^2;

(iii) $(x+1)^2$.

(a) Let $a, b, c \in \mathbb{Q}$ s.t.

$a(x+1) + b(x^3+x) + c(2x^3+1) = \vec{0}$

$\Rightarrow ax + a + bx^3 + bx + c2x^3 + c = \vec{0}$

$(b+2c)x^3 + (a+b)x + (a+c) = \vec{0}$

$\Rightarrow \begin{cases} b+2c = 0 \\ a+b = 0 \\ a+c = 0 \end{cases} \Rightarrow \begin{cases} b = -2c \\ a = -b \\ a = -c \end{cases} \Rightarrow \begin{cases} b = -2c \\ b = c \end{cases} \Rightarrow \{ c = -2c$

$c = -2c$ only when $c = 0$

$\Rightarrow a = 0, b = 0$

$\Rightarrow \exists$ only the trivial representation of $\vec{0}$

$\Rightarrow$ S lin. independent $\square$

(b) (i) $3x^3+3$

$\Rightarrow \begin{cases} b+2c = 3 \\ a+b = 0 \\ a+c = 3 \end{cases} \Rightarrow \begin{cases} b+2c = 3 \\ a = -b \\ -b+c = 3 \end{cases} \Rightarrow \begin{cases} b = -2c+3 \\ a = -b \\ b = c-3 \end{cases} \Rightarrow \begin{cases} c-3 = -2c+3 \\ a = -b \end{cases}$

$\Rightarrow \begin{cases} c = 2 \\ b = -1 \\ a = 1 \end{cases} \Rightarrow \exists$ lin comb. $= 3x^3+3 \Rightarrow 3x^3+3 \in \text{span}(S)$ $\square$

(ii) x^2

There is no x^2 term in any of span (S) $\Rightarrow x^2 \notin \text{span}(S)$ $\square$

(iii) $(x+1)^2 = x^2+2x+1$

There is no x^2 term in any of span (S) $\Rightarrow (x+1)^2 \notin \text{span}(S)$ $\square$

Problem 4. *10pts.*

Let V be a vector space over a field F, and let $S \subseteq V$ be a subset of V. Suppose that $v \in \operatorname{span} S$ and $w \notin \operatorname{span} S$. Prove that $v + w \notin \operatorname{span} S$.

span (S) subspace of V

Pf. by contradiction

Suppose $v \in \text{span}(S)$, $w \notin \text{span}(S)$.

and $v+w \in \text{span}(S)$.

$\exists\ a_1x_1 + \dots + a_nx_n = v+w$, where $x_1, \dots, x_n \in S$
$a_1, \dots, a_n \in \mathbb{F}$.

Since $v \in \text{span}(S)$, $\exists\ b_1x_1 + \dots + b_nx_n = v$, where $x_1, \dots, x_n \in S$
$b_1, \dots, b_n \in \mathbb{F}$.

$\Rightarrow a_1x_1 + \dots + a_nx_n = b_1x_1 + \dots + b_nx_n + w$

$w = (a_1 - b_1)x_1 + \dots + (a_n - b_n)x_n$

$\Rightarrow w \in \text{span}(S)$.

This is a contradiction. $\square$

Problem 1. Prove by induction that for every natural number $n \geq 1$

$$1 \cdot 2 + 2 \cdot 2^2 + \ldots + n \cdot 2^n = 2(1 + (n-1) \cdot 2^n)$$

The base case: When $n = 1$, the left hand side becomes $1 \cdot 2 = 2$, and the right hand side becomes $2(1 + (1-1) \cdot 2^1) = 2$.

The inductve case: Suppose that

$$1 \cdot 2 + 2 \cdot 2^2 + \ldots + n \cdot 2^n = 2(1 + (n-1) \cdot 2^n)$$

Then

$$\begin{aligned} 1 \cdot 2 + \ldots + (n+1) \cdot 2^{n+1} &= 2\,(1 + (n-1) \cdot 2^n) + (n+1) \cdot 2^{n+1} \\ &= 2\,(1 + (n-1) \cdot 2^n + (n+1) \cdot 2^n) \\ &= 2\,(1 + (2n) \cdot 2^n) = 2\left(1 + n \cdot 2^{n+1}\right) \\ &= 2\left(1 + ((n+1) - 1) \cdot 2^{n+1}\right) \end{aligned}$$

Problem 2. Let $V = \mathcal{F}(\mathbf{R}, \mathbf{R})$ be the vector space over the field $\mathbf{R}$ of all functions from $\mathbf{R}$ to $\mathbf{R}$.

a. Prove that the following subset is a subspace of V:

$$W = \{f \in V : f(\pi) = f(e) = 0\}$$

Suppose $f, g \in V$, and $\lambda \in \mathbf{R}$ Then $f(\pi) = f(e) = 0$ and $g(\pi) = g(e) = 0$. So $(f+g)(\pi) = f(\pi) + g(\pi) = 0$ and $(f+g)(e) = f(e) + g(e) = 0$. Similarly, $\lambda f(\pi) = \lambda f(e) = \lambda \cdot 0 = 0$. Hence V is closed under addition and scalar multiplication.

b. Prove that the following is **not** a subspace of V:

$$Z = \{f \in V : f(\pi) = f(e) = 1\}$$

The constant function $f = 1$ is in V. But $f + f$ is not.

Problem 3. Let $V = P_3(\mathbf{Q})$ be the vector space of polnomials with coëfficients in $\mathbf{Q}$, of degree at most 3. Let $S = \{x+1, x^3+x, 2x^3+1\}$.

a. Is S a linearly dependent set or not? Iustify your answer.

It is linearly independent.

Suppose $a(x+1)+b(x^3+x)+c(2x^3+1)=0$. Then

$$(a+2c)x^3+(a+b)x+(a+c)=0$$

Hence

$$\begin{aligned} a+2c &= 0 \\ a+b &= 0 \\ a+c &= 0 \end{aligned}$$

We can solve these equations to get $a=b=c=0$.

b. For each of the following polynomials, determine whether on not they belong to span$S$. Iustify all our answers.

(i) $3x^3+3$

Yes. $1(x+1)+-1(x^3+x)+2(2x^3+1)=3x^3+3$

(ii) x^2

No. The coëfficient of x^2 in each polynomial in S is 0.

(iii) $(x+1)^2 = x^2+2x+1$

No. You can never get a nonzero coëfficient for the x^2 term.

Problem 4. Let V be a vector space over a field F and let $S \subset V$ be a subset of V; suppose that $v \in \text{span}S$ and $w \notin \text{span}S$. Prove that $v+w \notin \text{span}S$.

If $v+w$ were in span$S$, then $-v+v+w$ would be as well, for span$S$ is a subspace of V. But this is equal to w. So $v+w$ cannot be in span$S$.