University of California, Los Angeles
Spring 2021

Instructor: T. Arant
Date: May 21, 2021

Name:________________ UCLA ID:________ Signature:________

MATH 115A: Linear Algebra
Midterm 2

This exam contains 7 pages (including this cover page) and 4 problems. Your solutions to the problems must be uploaded to Gradescope before 8am PST on May 22, 2021.

This is a take-home exam. The following rules regarding the take-home format apply:

- The exam is open-book/open-notes/open-internet.

  You **cannot** collaborate in any way with any individual on the exam. Any form of communication/consultation/collaboration with another person about the exam is expressly prohibited—this includes, but is not limited to, Zoom meetings, email, telephone calls, texting, making posts on stack exchanges, etc. Violation of the no-collaboration policy is a violation of the UCLA code of student conduct and will come with serious consequences.

- The instructor reserves the right to ask any student for clarification regarding any of the student's exam answers at any time during a two week period after the day of the exam. This may require a Zoom meeting with the instructor.

- Please sign the pledge on the next page and upload an image of the signed pledge onto Gradescope when uploading your exam.

You are required to show your work on each problem of this exam. The following rules apply:

- **All answers must be justified. Mysterious or unsupported answers will not receive credit**. A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- **If you use a theorem or proposition from class or the notes or the textbook or a result established in the homework, you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

Good luck!

Academic integrity pledge:

Upon my honor, I affirm that I did not solicit nor did I receive the help of any individual in writing my answers to this exam.

Signature: ______________________________

Print name: ______________________________

1. In each part, determine whether or not a linear transformation satisfying the given description exists. If such a transformation exists, give an example of one such transformation and verify that it has the desired properties. If no such transformations exist, provide a proof that none exist.

   (a) (5 points) A linear $T : F^3 \to F^3$ such that $N(T) \subseteq R(T)$ and T is not one-to-one.

   (b) (5 points) An onto linear transformation $T : F^3 \to P_2(F)$ such that $T(e_1) = 1 + X$ and $T(e_2) = 2 + 2X$.

2. Let $T : P_2(F) \to F^3$ be the unique linear transformation satisfying

$$T(1) = (1, 1, 1), \quad T(X) = (1, 0, -1), \quad T(X^2) = (2, 1, 0).$$

(a) (5 points) Let $\beta = \big(1, 1 + X, 1 + X - X^2\big)$ and $\gamma = \big(e_1, e_2, e_3\big)$, which are ordered bases of $P_2(F)$ and F^3, respectively. Compute $[T]_\beta^\gamma$.

(b) (5 points) Find a basis for $R(T)$. Provide justification that it is indeed a basis of $R(T)$.

(c) (5 points) Let $\delta = \big(e_3, e_2, e_1\big)$. Compute the change of basis matrix $[I_{F^3}]_\gamma^\delta$ and use it to compute $[T]_\beta^\delta$.

3. (10 points) Let V and W be finite dimensional vector spaces over F such that $\dim(V) \leq \dim(W)$.

   Use the linear transformation construction theorem to prove that there exists a one-to-one linear transformation $T : V \to W$.

4. (10 points) Let V be a finite dimensional vector space over F of dimension n. Let $T : V \to V$ be a linear operator on V with $\text{nullity}(T) = k \le n$. Prove there exists an ordered basis β of V such that

$$[T]_\beta = \begin{bmatrix} O & B \end{bmatrix},$$

where O is the $n \times k$ zero matrix and $B \in M_{n\times(n-k)}(F)$ is a matrix such that each column of B is a nonzero vector of F^n.