University of California, Los Angeles
Spring 2021

Instructor: T. Arant
Date: April 28, 2021

Name:____________________________ UCLA ID:____________ Signature:____________

# MATH 115A: Linear Algebra
# Midterm 1

This exam contains 6 pages (including this cover page) and 4 problems. Your solutions to the problems must be uploaded to Gradescope before 8am PST on April 29, 2021.

This is a take-home exam. The following rules regarding the take-home format apply:

- The exam is open-book/open-notes/open-internet.

  You **cannot** collaborate in any way with any individual on the exam. Any form of communication/consultation/collaboration with another person about the exam is expressly prohibited—this includes, but is not limited to, Zoom meetings, email, telephone calls, texting, making posts on stack exchanges, etc. Violation of the no-collaboration policy is a violation of the UCLA code of student conduct and will come with serious consequences.

- The instructor reserves the right to ask any student for clarification regarding any of the student's exam answers at any time during a two week period after the day of the exam. This may require a Zoom meeting with the instructor.

- Please sign the pledge on the next page and upload an image of the signed pledge onto Gradescope when uploading your exam.

You are required to show your work on each problem of this exam. The following rules apply:

- **All answers must be justified. Mysterious or unsupported answers will not receive credit**. A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- **If you use a theorem or proposition from class or the notes or the textbook or a result established in the homework, you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- When applicable, it is acceptable (and even preferable) for your final answers to contain unsimplified terms of the form $n^m$, $n!$, ${}_nP_r$ and ${}_nC_r$

Good luck!

Academic integrity pledge:

Upon my honor, I affirm that I did not solicit nor did I receive the help of any individual in writing my answers to this exam.

Signature: ____________________________

Print name: ____________________________

1. We define a candidate vector space structure over $F$ as follows. Let $V = P_2(F)$, the set of degree at most 2 polynomials with coefficients in $F$. A scalar mutliplication operation is defined as follows: for $c \in F$ and $a_0 + a_1X + a_2X^2 \in V$,

$$c(a_0 + a_1X + a_2X^2) = ca_0 + ca_1X.$$

We define an addition operation as follows: for $a_0 + a_1X + a_2X^2,\ b_0 + b_1X + b_2X^2 \in V$,

$$(a_0 + a_1X + a_2X^2) + (b_0 + b_1X + b_2X^2) = (a_0 + b_0) + (a_1 + b_1)X.$$

(a) (5 points) Prove that this structure satisfies the vector space axiom

(VS 7) for all $u, v \in V$ and all $c \in F$, $c(u + v) = cu + cv$.

(b) (5 points) Show that this structure does **not** satisfy vector space axiom (VS 3), i.e., show that no element of $V$ is the zero vector for the + operation defined above.

2. (10 points) Let $n$ be a postive integer, let $V = M_{n\times n}(F)$ be the vector space of $n$ by $n$ matrices over $F$. Suppose $W \subseteq V$ is a subspace of $V$. Define $U = \{A \in V : A^t \in W\}$.

   Prove that $U$ is a subspace of $V$.

3. Let $V$ be a vector space over $F$. Let $v_1, \ldots, v_k, u_1, \ldots, u_m \in V$ be distinct vectors.

   (a) (5 points) Assume $\{v_1, \ldots, v_k\}$ is linearly independent and that $\{v_1, \ldots, v_k, u_1, \ldots, u_m\}$ is linearly dependent. Prove that if $a_1, \ldots, a_k, b_1, \ldots, b_m \in F$ give a non-trivial solution for the zero equation,

   $$a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_m u_m = 0,$$

   then at least one $b_i$, $1 \leq i \leq m$, is non-zero.

   (b) (5 points) Assume now $k = m$ and that $\{v_1, \ldots, v_k, u_1, \ldots, u_k\}$ is linearly independent. Prove that $\{v_1 + u_1, \ldots, v_k + u_k\}$ is linearly independent.

4. Let $V = M_{2\times 2}(F)$.

   (a) (6 points) Prove that there exists subspaces $U, W \subseteq V$ such that $\dim(U) = 3$, $\dim(W) = 1$, and $V = U \oplus W$.

   (b) (4 points) Do subspaces $U, W \subseteq V$ exist such that $\dim(U) = 3$, $\dim(W) = 2$ and $V = U \oplus W$? Prove your answer.