University of California, Los Angeles
Spring 2021

Instructor: T. Arant
Date: June 8, 2021

Name:________________ UCLA ID:________ Signature:________

MATH 115A: Linear Algebra
Final Exam

This exam contains 13 pages (including this cover page) and 6 problems. Your solutions to the problems must be uploaded to Gradescope before 8am PST on June 9, 2021.

This is a take-home exam. The following rules regarding the take-home format apply:

- The exam is open-book/open-notes/open-internet.

  You **cannot** collaborate in any way with any individual on the exam. Any form of communication/consultation/collaboration with another person about the exam is expressly prohibited—this includes, but is not limited to, Zoom meetings, email, telephone calls, texting, making posts on stack exchanges, etc. Violation of the no-collaboration policy is a violation of the UCLA code of student conduct and will come with serious consequences.

- The instructor reserves the right to ask any student for clarification regarding any of the student's exam answers at any time during a two week period after the day of the exam. This may require a Zoom meeting with the instructor.

- Please sign the pledge on the next page and upload an image of the signed pledge onto Gradescope when uploading your exam.

You are required to show your work on each problem of this exam. The following rules apply:

- **All answers must be justified. Mysterious or unsupported answers will not receive credit**. A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- **If you use a theorem or proposition from class or the notes or the textbook or a result established in the homework, you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

Good luck!

Academic integrity pledge:

Upon my honor, I affirm that I did not solicit nor did I receive the help of any individual in writing my answers to this exam.

Signature: ____________________

Print name:____________________

Important conventions for the entire exam:

Assume every vector space and every inner product space in the exam is finite dimensional.

F is either $\mathbb{R}$ or $\mathbb{C}$.

1. Each part of this question gives a mathematical statement, and you will determine whether the statement is true or false. If the statement is true, clearly indicate this and provide a proof that the statement is true. If the statement is false, clearly indicate this and provide a proof that the statement is false.

   (a) (5 points) Let $S = \{v_1, \ldots, v_n\}$ be a set of n vectors in a vector space V. If S is linearly dependent, then there exists $1 \leq i, j \leq n$, $i \neq j$, such that v_i is a scalar multiple of v_j.

   (b) (5 points) There is a linear $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x)$ is orthogonal to x for every $x \in \mathbb{R}^2$.

(c) (5 points) If $T, U : V \to V$ are simultaneously diagonalizable linear operators on a vector space V, then $TU : V \to V$ is diagonalizable.

(d) (5 points) If $\dim(V) \leq \dim(W)$, then there is a subspace $Z \subseteq W$ which is isomorphic to V.

2. Each part of this question will ask you to compute something. Show work that justifies your computation.

   (a) (5 points) Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable and has exactly $n-1$ many distinct eigenvalues $\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1} \in \mathbb{R}$. Also assume $\mathrm{GM}_T(\lambda_i) = 1$ for $i = 1, \ldots, n-2$. Compute $\mathrm{AM}_T(\lambda_{n-1})$.

   (b) (5 points) Let $V = M_{2\times 2}(F)$ and let $T : V \to V$ be the projection onto $W_1 = \{A \in V : A^t = A\}$ along $W_2 = \{A \in V : A^t = -A\}$.
   Let $\beta = (E^{11}, E^{22}, E^{12} + E^{21}, E^{12} - E^{21})$. Compute $[T]_\beta$.

(c) (5 points) Let V be a dimension 7 inner product space, and let $W_1, W_2 \subseteq V$ be subspaces with $\dim(W_1) = \dim(W_2) = 3$ and $\dim(W_1 \cap W_2) = 2$. Compute $\dim(W_1^{\perp} \cap W_2^{\perp})$.

(d) (5 points) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear transformation such that $\{T(e_1), T(e_2), T(e_2)\}$ is an orthogonal set of 3 vectors. Compute $\text{nullity}(T)$.

3. (10 points) Let $T : V \to W$ be alinear transformation, and let $x_1, \dots, x_k, y \in V$. Prove that if $\{T(x_1), \dots, T(x_k)\}$ is a linearly independent set of k many vectors and $y \in N(T)$, then $S = \{x_1, \dots, x_k, y\}$ is a linearly independent set of $k+1$ many vectors.

4. (10 points) Let $T_1 : V \to W$ and $T_2 : W \to Z$ be linear transformations. Let $U : V \to Z$ be the zero transformation.

   Prove that $T_2T_1 = U$ if and only if $R(T_1) \subseteq N(T_2)$.

5. Define $T : F^n \to F^n$ by $T(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, 0)$. You may use without justification that T is linear.

   (a) (5 points) Compute the characteristic polynomial of T.

   (b) (6 points) Show that T has only one eigenvalue and find the eigenspace of this one eigenvalue.

(c) (4 points) Is T diagonalizable? Justify your answer.

6. Let V be an inner product space. Suppose $w_1, w_2 \in V$ are unit vectors which are orthogonal to each other. Let $w_3 = w_1 + w_2$.

   (a) (4 points) Compute the orthogonal projection of the vector w_3 onto the subspace $\operatorname{span}\{w_1\}$.

   (b) (6 points) Let $S' = \{v_1, v_2, v_3\}$ be the set obtained by applying the Gram-Schmidt process to the set $S = \{w_1, w_2, w_3\}$. Use the formulas from the Gram-Schmidt process to write each of v_1, v_2, and v_3 as linear combinations of vectors in S.

(c) (5 points) The set S' obtained from part (a) is not an orthogonal set of nonzero vectors. Explain why this does not contradict the theorem we proved in lecture about the Gram-Schmidt process (Theorem 69, Lecture Notes part 3).