Solutions to practice problems

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Problem 0.0.1. Let $W_1 = \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$ and $W_2 = \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ Give a basis for W_1, W_2, $W_1 \cap W_2$ and $W_1 + W_2$ and calculate $\dim W_1 + \dim W_2 - \dim W_1 \cap W_2$.

Solution We see that the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are linearly independent and form a basis for W_1. A slightly more convenient basis is given by the two vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. On the other hand we have

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

hence those four don't form a basis for W_2, but the first three are clearly linearly independent and form a basis. For $W_1 \cap W_2$ we check that the only linear combination of basis vectors in W_2 that is also in W_1 are multiples of $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (this is because the other basis vectors all have 1's in places where all vectors in W_1 have 0's). Hence this one vector is a basis for $W_1 \cap W_2$.

For $W_1 + W_2$ we use the formula

$$\text{span} S_1 + \text{span} S_2 = \text{span}(S_1 \cup S_2)$$

throw out the double vector and see that a basis for it is given by the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. So we get

$$\dim W_1 + \dim W_2 - \dim W_1 \cap W_2 = 2 + 3 - 1 = 4 = \dim(W_1 + W_2).$$

Problem 0.0.2. Show that for a linear transformation $T : V \to W$ the nullity $N(T)$ and the range $R(T)$ are subspaces of V and W respectively.

Solution This is a Theorem in the book. The proof is also to be found there.

Problem 0.0.3. Prove in the following that T is a linear transformation and give a basis for $N(T)$ and $R(T)$.

- $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(a_1, a_2, a_3) := (a_1 - a_2, a_3)$.
- $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ with $T(f(x)) = xf(x) + f'(x)$.

Solution For the first part proving linearity is straight-forward. To find the nullity we need to solve $T(a_1, a_2, a_3) = 0$ for a_1, a_2 and a_3. Inserting the definition yields $a_1 - a_2 = 0$ and $2a_3 = 0$. So the nullity space is the space of all vectors with $a_1 = a_2$ and $a_3 = 0$. This is clearly spanned by $(1, 1, 0)$ and hence this one vector is a basis. Now looking at the rank-nullity theorem I know that the range needs to hae dimension 2 and hence I get $R(T) = \mathbb{R}^2$ a basis of which is $(1, 0)$ and $(0, 1)$.

For the second part we first check linearity: Let $a \in \mathbb{R}$ be a scalar and $f(x), g(x) \in P_2(\mathbb{R})$ polynomials. We have

$$T(af(x) + g(x)) = axf(x) + xg(x) + af'(x) + g'(x) = aT(f(x)) + T(g(x))$$

and T is linear. To find the nullity we solve $T(f(x)) = 0$. It is helpful to set up $f(x) = a_2x^2 + a_1x + a_0$. Then the nullity space is the space of all $f(x)$ where

$$a_2x^3 + a_1x^2 + a_0x + 2a_2x + a_1 = 0.$$

Looking at the coefficients of x^3 we get $a_2 = 0$ and then looking at the coefficients of x^2 we get $a_1 = 0$ and finally $a_0 = 0$. Hence $N(T) = 0$ and T is one-to-one. Therefor a the image of a basis of the source is a basis of the range. Since $\{x^2, x, 1\}$ is a basis for $P_2(\mathbb{R})$ we see that $\{x^3 - 2x, x^2 - 1, x\}$ is a basis for $R(T)$.

Problem 0.0.4. Prove Let $f(x) \in P_n(\mathbb{R})$ be a polynomial of degree n. Show that for all $g(x) \in \mathcal{P}_n(\mathbb{R})$ there exist unique $c_0, \ldots, c_n \in \mathbb{R}$ such that

$$g(x) = c_0f(x) + c_1f'(x) + \ldots + c_nf^{(n)}(x)$$

Proof This statement is equivalent to saying that the $f(x), f'(x), \ldots f^{(n)}(x)$ form a basis of $P_n(\mathbb{R})$. We know that $\dim P_n(\mathbb{R}) = n+1$ and the $f^{(}i)(x)$'s are $n+1$ different vectors. So it is enough to show that they are linearly independent. Again, it helps to write $f(x) = a_n x^n + \ldots + a_0$ with $a_n \neq 0$. Now we see that $f(x)$ is the only polynomial of the above that has a nonzero coefficient at x^n. So in any linear combination of the $f^{(i)}(x)$ that yields 0, the coefficient of $f(x)$ must be 0. Moving one down we can now see that the coefficient of $f'(x)$ in such a linear combination must also be 0 as $f(x)$ and $f'(x)$ are the only two polynomials in our list with a nonzero coefficient in front of x^{n-1}. Playing this game to all the way down (or, more formally, using induction) I can prove that those polynomials are linearly independent.