MATH 115A SPRING 2021 MIDTERM 1 SOLUTIONS

Problem 1: Let V be a vector space over a field $\mathbb{F}$.

(a) Prove that for every $v \in V$ and $a \in \mathbb{F}$, if $av = \vec{0}$, then either $a = 0$ or $v = \vec{0}$, where 0 is the additive identity in $\mathbb{F}$ and $\vec{0}$ is the additive identity in V.

(b) Deduce that if there exists a nonzero vector $v \in V$ such that $v = -v$, then the characteristic of $\mathbb{F}$ is 2.

Solution: We will prove the contrapositive of the statement in part (a). Suppose $a \neq 0$ and $v \neq \vec{0}$. Assume for sake of contradiction that $av = \vec{0}$. Then, since $a \in \mathbb{F}$ is nonzero, there exists a^{-1} such that

$$a^{-1}a = 1.$$

So, we have

$$\vec{0} = a^{-1}(av) = (a^{-1}a)v = v$$

a contradiction. Hence, if both a and v are nonzero, so is av. Hence, by proof by contrapositive, if $av = \vec{0}$, then either $a = 0$ or $v = \vec{0}$.

For part (b), note that if $v = -v$, we can add v to both sides of the equation to get $2v = \vec{0}$. Since v is nonzero, by part (a), we must have $2 = 0$ in $\mathbb{F}$. Since the characteristic is the smallest positive integer that is 0 in the field, and $1 \neq 0$, $\mathbb{F}$ must have characteristic 2.

Problem 2: Fix some integer $n \geq 0$ and some field $\mathbb{F}$. For each i from 0 to n, let us pick some polynomial $p_i(x)$ in $P_n(\mathbb{F})$ of degree i. Prove that

$$\{p_0(x), p_1(x) \ldots, p_n(x)\}$$

is a basis for $P_n(\mathbb{F})$.

Solution: Here I provide two solutions. In both methods, we use the fact that $P_n(]\mathbb{F})$ has a standard basis $\{1, \ldots, x^n\}$ and is hence dimension $n+1$. Since the given set has size $n+1$, we just need to prove it is linearly independent.

In the first proof, we use mathematical induction to prove that $\{p_0, \ldots, p_i\}$ is linearly independent for all $i \geq 0$. The base case of $i = 0$ is trivial as p_0 is a nonzero constant, as it has degree 0, and $\{p_0\}$ is hence linearly independent. So, assume as inductive hypothesis that $\{p_0, \ldots, p_{i-1}\}$ is linearly independent. p_i is not in the span of $\{p_0, \ldots, p_{i-1}\}$, as these are polynomials of degree $< i$, while p_i has degree i. Hence,

$$\{p_0, \ldots, p_i\}$$

is also linearly independent.

In the second proof, we use a proof by contradiction. Again, we want to prove $\{p_0, \ldots, p_n\}$ is linearly independent. So, suppose we have a linear relation

$$a_0p_0 + \cdots + a_np_n = 0$$

and assume for sake of contradiction that not all of the a_i are 0. Let m be the biggest integer such that $a_m \neq 0$. Then, we can remove the terms involving the higher degree polynomials and get

$$a_0p_0 + \cdots + a_mp_m = 0.$$

Now, if $p_m = b_mx^m + \cdots + b_0$, then as it has degree m, $b_m \neq 0$. Hence, in the linear combination

$$a_0p_0 + \cdots + a_mp_m$$

the coefficient of x^m is a_mb_m, as the polynomials $p_0, \ldots, p_{m-1}$ have degree $< m$. Since a_m and b_m are both nonzero, so is a_mb_m, which contradicts the relation

$$a_0p_0 + \cdots + a_mp_m = 0.$$

Hence, $\{p_0, \ldots, p_n\}$ must be linearly independent.

Problem 3: For each of the following problems, either provide the requested example with justification, or prove that it does not exist.

(a) Four vectors v_1, v_2, v_3, v_4 in $\mathbb{R}^4$ and a subspace W of dimension 2 of $\mathbb{R}^4$ such that $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$ and W does not contain v_1, v_2, v_3, v_4.

(b) Two subspaces of $\mathbb{R}^4$ of dimension 3 whose intersection is the trivial subspace.

(c) Two subspaces of $\mathbb{R}^4$ of dimension 2 whose intersection is the trivial subspace.

(d) A linearly dependent set $\{v_1, v_2, v_3\}$ in $\mathbb{R}^3$ consisting of three distinct, nonzero vectors v_i such that each v_i is in the span of the other two vectors.

Solution:

(a) Let v_1, v_2, v_3, v_4 be the standard basis vectors $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ respectively. Let W be the subset $\{(x, y, z, w) : x = y, z = w\}$. This is a subspace because if we have two vectors of the form

$$v_1 = (x, x, z, z), v_2 = (x', x', z', z')$$

then, for any $c \in \mathbb{F}$,

$$v_1 + cv_2 = (x + cx', x + cx', z + cz', z + cz') \in W$$

as well. It also dimension 2, as $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$ is a basis for W. Finally, it clearly does not contain any of the standard basis vectors.

(b) Two subspaces of $\mathbb{R}^4$ of dimension 3 cannot intersect trivially. Suppose for sake of contradiction, that we did have V, W subspaces of $\mathbb{R}^4$ that intersected in the trivial subspace. Let $\{v_1, v_2, v_3\}$ be a basis for V and $\{w_1, w_2.w_3\}$ be a basis for W. Then, we claim that $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ is linearly independent. If we did have a relation

$$a_1v_1 + a_2v_2 + a_3v_3 + b_1w_1 + b_2w_2 + b_3w_3 = 0$$

with not all the coefficients equal to 0, then we get

$$v = a_1v_2 + a_2v_2 + a_3v_3 = -b_1w_1 - b_2w_2 - b_3w_3 \in V \cap W$$

and v cannot be 0 as $\{v_1, v_2, v_3\}$ is linearly independent. This is a contradiction, as we now get a linearly independent subset of $\mathbb{R}^4$ of size 6.

(c) Take $V = \text{Span}\{(1,0,0,0),(0,1,0,0)\}$ and $W = \text{Span}\{(0,0,1,0),(0,0,0,1)\}$. Then, V and W have dimension 2 and intersect trivially.

(d) Take $v_1 = (1,0,0)$, $v_2 = (2,0,0)$, $v_3 = (3,0,0)$. Each vector is a multiple of each of the other two vectors and hence satisfies the desired property.

Problem 4: Let $\mathbb{F}$ be a field and fix some nonzero vector v in $\mathbb{F}^5$. Our goal in this problem is to characterize the subspaces of $\mathbb{F}^5$ that contain v and have dimension 2.

(a) Let W be a subspace of $\mathbb{F}^5$ of dimension 2 that contains v. Prove that for any $w \in W$ that is not a scalar multiple of v, $\{v, w\}$ is a basis for W.

(b) Show that $\{v, w_1\}$ and $\{v, w_2\}$ are bases for the same two-dimensional subspace that contains v if and only if $w_2 = av + bw_1$ for $a, b \in \mathbb{F}$, with $b \neq 0$.

Solution: Let us fix our nonzero vector v.

(a) Let W be a subspace of dimension 2 containing v. Let $w \in W$ be any vector that is not a scalar multiple of v. Then, w is not in $\text{Span}\{v\}$ and hence $\{v, w\}$ is linearly independent. Since W has dimension 2, any linearly independent subset of W of size 2 is a basis. Hence, $\{v, w\}$ is a basis.

(b) Suppose $\{v, w_1\}$ and $\{v, w_2\}$ are both bases for the same 2-dimensional subspace W. Then, w_2 is in the span of $\{v, w_1$ and hence

$$w_2 = av + bw_1$$

for some $a, b \in \mathbb{F}$. b cannot be zero as w_2 is not a scalar multiple of v. Conversely, suppose we have $\{v, w_1\}$ a basis for a 2-dimensional subspace W and

$$w_2 = av + bw_1$$

for $a, b \in \mathbb{F}$ with b nonzero. Then, w_2 is in W as well as it is a linear combination of vectors in W. Furthermore, as $\{v, w_1\}$ is linearly independent, w_1 is not a scalar multiple of v. Hence, $\{v, w_2\}$ is linearly independent of size equal to the dimension of W and is hence a basis for W as well.