Construct an isomorphism
$T: V \to F^3$.
① Write down the T function.
② T is linear
③ 1-1
④ on-to
⑤ $\dim V = \dim F^3$
or construct $S: F^3 \to V$
and show $S \circ T = Id|_V$
$T \circ S = Id|_{F^3}$

1 Let

$$V = \left\{ \begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix} : a, b, c \in F \right\}.$$

Construct an isomorphism from V to F^3.

$V \xrightarrow{T} F^3$

for $\forall v \in V$, and $v = \begin{bmatrix} c_1 & c_1+c_2 \\ 0 & c_3 \end{bmatrix} : c_1, c_2, c_3 \in F$.

$T(v) = c_1 e_1 + (c_1+c_2)e_2 + c_3 e_3$ and $\{e_1, e_2, e_3\}$ is an ordered basis for F^3.

① for v and $u \in V$,
$v = \begin{bmatrix} c_1 & c_1+c_2 \\ 0 & c_3 \end{bmatrix}$ and $u = \begin{bmatrix} a_1 & a_1+a_2 \\ 0 & a_3 \end{bmatrix}$

$T(dv+u) = T\left(\begin{bmatrix} dc_1+a_1 & d(c_1+c_2)+(a_1+a_2) \\ 0 & dc_3+a_3 \end{bmatrix}\right) = (dc_1+a_1)e_1 + [d(c_1+c_2)+(a_1+a_2)]e_2 + (dc_3+a_3)e_3$

because of distributivity of vectors
$= dc_1e_1 + a_1e_1 + dc_1e_2 + dc_2e_2 + a_1e_2 + a_2e_2 + dc_3e_3 + a_3e_3$
$= d[c_1e_1 + c_1e_2 + c_2e_2 + c_3e_3] + [a_1e_1 + a_1e_2 + a_2e_2 + a_3e_3]$
$= d[c_1(e_1) + (c_1+c_2)e_2 + c_3e_3] + [a_1e_1 + (a_1+a_2)e_2 + a_3e_3]$

From definition I gave,
$T(v) = c_1e_1 + (c_1+c_2)e_2 + c_3e_3$
and $T(u) = a_1e_1 + (a_1+a_2)e_2 + a_3e_3$
so $T(dv+u) = dT(v) + T(u)$.

Therefore, T is a linear transformation. □

② Assume $T(\bar{v}) = T(\bar{u})$.
$T(\bar{v}) = c_1e_1 + (c_1+c_2)e_2 + c_3e_3$
and $T(\bar{u}) = a_1e_1 + (a_1+a_2)e_2 + a_3e_3$

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2 Define the linear transformation $T : P_2(R) \to M_{2\times 2}(R)$ by

$$T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

a Using $\beta = \{1, x, x^2\}$ as a basis for $P_2(R)$, what is the basis for the image of T, $R(T)$?

From book, we know

$R(T) = \text{span}\{\beta\} = \text{span}\{T(1), T(x), T(x^2)\}$.

$T(1) = \begin{pmatrix} 1-1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1-2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

$T(x^2) = \begin{pmatrix} 1-4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore R(T) = \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}$.

From observation, $\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} = 3\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, so

$\text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ becau[se]

we can eliminate the dependent vector from theo[rem]

so the basis for $R(T)$ is $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$.

b What is the dimension of $R(T)$?

Dimension of $R(T)$ is equal to number of basis $R(T)$ has.

for $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$, for any $c_1, c_2 \in F$

$c_1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_2\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} -c_2 & 0 \\ 0 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so $\begin{cases} -c_2 = 0 \\ c_1 = 0 \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases}$

[illegible] definition $\begin{pmatrix} 0 & 0 \\ \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ \end{pmatrix}$ is

3 Let $T : R^2 \to R^3$ be the linear transformation defined by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$. Let $\beta = (e_1, e_2)$ be the standard basis for R^2 and let $\gamma = (e_1, e_2, e_3)$ be the standard basis for R^3. Find the matrix representation of T, $[T]_\beta^\gamma$.

$$T(e_1) = T(1,0) = (1+0, 0, 2-0) = (1, 0, 2)$$

$$= 1 \cdot e_1 + 0 \cdot e_2 + 2e_3$$

$$T(e_2) = T(0, 1)$$

$$= (0+3, 0, 0-4)$$

$$= (3, 0, -4)$$

$$= 3 \cdot e_1 + 0 \cdot e_2 + (-4) e_3$$

$$\therefore [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$$

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4 Let $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$ be two bases for $P_2(F)$. Find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

β-coordinates into β'-coordinates

$\beta' \to \beta$

we want to find $[I]_{\beta'}^{\beta}$ first

what is this notation

$I(a_2x^2 + a_1x + a_0) = a_2x^2 + a_1x + a_0$

$= a_2 \cdot x^2 + a_1 \cdot x + a_0 \cdot 1$

$I(b_2x^2 + b_1x + b_0) = b_2x^2 + b_1x + b_0$

$= b_2 \cdot x^2 + b_1 \cdot x + b_0 \cdot 1$

$I(c_2x^2 + c_1x + c_0) = c_2x^2 + c_1x + c_0$

$= c_2 \cdot x^2 + c_1 \cdot x + c_0 \cdot 1$

$$[T]_{\beta'}^{\beta} = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{bmatrix} \checkmark$$

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$$Q = I^{-1} \cdot [T]_{\beta'}^{\beta} \cdot I$$

$$= \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{bmatrix}$$

5 Let V be a vector space over F, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

$T(T v) = T_0 v = 0$
$T(w) = 0$
$w \in N(T)$

$T: V \to V$

$\Rightarrow$ Assume $T^2 = T_0$.

$\forall \bar{v} \in V$, $\bar{w} = T(\bar{v}) \in R(T)$.

$T^2 \bar{v} = T(T(v)) = T(\bar{w})$

$T^2 \bar{v} = T_0 \bar{v} = 0$

$\therefore T(\bar{w}) = 0$.

therefore $\bar{w} \in N(T)$.

this implies that for all vectors in $R(T)$, they also are in $N(T)$.

so $R(T) \subseteq N(T)$. $\square$

$\Leftarrow$ Assume $R(T) \subseteq N(T)$.

Since $R(T) \subseteq N(T)$,

so $\forall \bar{w} \in R(T)$, $\bar{w} \subseteq N(T)$.

since T is a linear operator,

so $\bar{w} \in V$.

so $T\bar{w} = 0$.

since $\bar{w} \in R(T)$, $\bar{w} = T(\bar{v})$ for $\forall \bar{v} \in V$,

therefore $T\bar{w} = T(T(\bar{v})) = 0$.

thus $T(T(\bar{v})) = 0$

$T^2(\bar{v}) = 0 = T_0$, because TU is also l

write in complete sentences.

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