Midterm 1

wonderful!
100

Math 115A

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Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers whenever possible to ensure full credit.
- Proofs should be in complete sentences.
  I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points,
- Good luck!

basis $\{1, x, x^2\}$

2 Define the linear transformation $T : P_2(R) \to M_{2\times 2}(R)$ by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

a Using $\beta = \{1, x, x^2\}$ as a basis for $P_2(R)$, what is the basis for the image of T, $R(T)$?

linear ind.

Plug in $\beta = \{1, x, x^2\}$ to T

$$T(1) = \begin{bmatrix} 1-1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 1-2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

These two are linearly independent, and therefore comprise the basis.

$$T(x^2) = \begin{bmatrix} 1-4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$$

Basis for $R(T)$:
$$\boxed{\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}}$$

b What is the dimension of $R(T)$?

Since there are two matrices in the basis, then the dimension of $R(T) = 2$.

3 Let $T : R^2 \to R^3$ be the linear transformation defined by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$. Let $\beta = (e_1, e_2)$ be the standard basis for R^2 and let $\gamma = (e_1, e_2, e_3)$ be the standard basis for R^3. Find the matrix representation of T, $[T]_\beta^\gamma$.

$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

Plug in values of β:

$T(e_1) = T(1, 0) = (1, 0, 2)$

$T(e_2) = T(0, 1) = (3, 0, -4)$

Matrix Representation $[T]_\beta^\gamma$: $\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$ ✓

4 Given a set of vectors S in a vector space V, show that $span(S)$ is a subspace of V.

Def:

- span(S) is all the possible linear combinations of the vectors of S in vector space V.

Let's say $S = \{v_1, v_2, \dots, v_n\}$

WTS: span(S) is subspace of V.

In order to prove that a set S is a subspace, we must show that S contains the zero element, and that it is closed under vector addition and scalar multiplication.

- Because the vectors S are contained in vsp V, then the zero element is also contained b/c V is a vsp, which means there must be some element 0 s.t. $\forall x \in V$, $x + 0 = x$. Also, by definition of span (above), then the span(S) is all linear combination s.t. an element $y = a_1v_1 + a_2v_2 + \dots + a_nv_2$ (for scalars $a_1, \dots, a_n$). Then if all scalars $a_1, \dots, a_n = 0$, then the zero element is contained in the span(S). ✓
- Closed under ⊕: By def. of span, two element x, y in span(S) is
  $y = a_1v_1 + \dots + a_nv_n$, & $x = b_1v_1 + \dots + b_nv_n$. (for scalars a_i, b_i, $1 \le i \le n$)

$x + y = a_1v_1 + \dots + a_nv_n + b_1v_1 + \dots + b_nv_n$
$= (a_1+b_1)v_1 + \dots + (a_n+b_n)v_n$. We can call $(a_1+b_1), \dots, (a_n+b_n)$ $c_1, \dots, c_n$, some scalars. In which case

$x + y = c_1v_1 + \dots + c_nv_n$ which is still a vector contained in the span(S). ∴ span(S) is closed under vector addition

- closed under scalar ⊗: Let $x = a_1v_1 + \dots + a_nv_n$ be an element in the span(S) and let c be some arbitrary scalar. Then we WTS cx is still contained in the span(S).

→ go to back side

5 In $M_{3\times 2}(F)$, prove that $\overset{A_1}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{A_2}{\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{A_3}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}}, \overset{A_4}{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{A_5}{\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}}$ are linearly dependant.

Def

A set $S = \{A_1, \dots, A_n\}$ is said to be <u>linearly dependent</u> when

$$0 = a_1 A_1 + \dots + a_n A_n \quad (*)$$

for some scalars $a_1, \dots, a_n$ (not all $= 0$).

Our given set $S = \{A_1, A_2, A_3, A_4, A_5\}$ is linearly dependent because

$$\underbrace{A_1 + A_2 + A_3}_{} - A_4 - A_5 = \underset{\substack{\uparrow \\ \text{zero element}}}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A + A_2 + 3 =$ $\qquad -A_4 - A_5 =$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Because this is a nontrivial solution to equation $(*)$ where $a_1, \dots, a_n$ are NOT all equal to zero, set $S = \{A_1, A_2 A_3 A_4, A_5\}$ is linearly dependent.

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