Take Home Midterm and Homework #7

Instructions: This homework/midterm is open book. All proofs must be fully written or typed out. You may not cut and paste solutions that you did not write or type yourself. If you copy part of or the whole proof from a reference you must cite that reference when used on that problem. If results not proven in class or homework are used, you must prove these also. You may not help each other or have others help solve the problems for you.

In the problems below, in which V is an **inner product space** you may assume that F equal to $\mathbb{R}$ or $\mathbb{C}$ (but you cannot assume which one unless it is given) with inner product $\langle\ ,\ \rangle_V$.

Problem 1. Let V be an inner product space. Prove that $d : V \times V \to \mathbb{R}$ defined by $d(v, w) = ||v - w||$ makes V into a **metric space**, i.e., for all $v, w, x \in V$, d satisfies all of the following:

(a) $d(v, w) \geq 0$ and equals zero if and only if $v = w$.

(b) $d(v, w) = d(w, v)$.

(c) (Triangle Inequality) $d(v, w) \leq d(v, x) + d(x, w)$.

Problem 2. Let V be a finite dimensional inner product space with an orthonormal basis $\{v_1, ..., v_n\}$. Let $v, w \in V$. Prove Parseval's Formula:

$$\langle v, w\rangle = \sum_{i=1}^{n} \langle v, v_i\rangle \overline{\langle w, v_i\rangle}.$$

In particular, the Pythagorean Theorem holds, viz.,

$$||v||^2 = \sum_{i=1}^{n} |\langle v, v_i\rangle|^2.$$

Problem 3. Suppose that V is the direct sum $V = W_1 \oplus W_2$ of vector spaces over F with $F = \mathbb{R}$ or $\mathbb{C}$. If W_i is an inner product space via $\langle\ ,\ \rangle_{W_i}$, $i = 1, 2$, show that there is a unique inner product $\langle\ ,\ \rangle_V$ on V satisfying

(a) $W_2 = W_1^{\perp}$.

(b) $\langle x, y\rangle_V = \langle x, y\rangle_{W_i}$, for $x, y \in W_i$, $i = 1, 2$.

Problem 4. Let V be a finite dimensional vector space over F (so F is any field) and $T : V \to V$ a linear operator.

(a) Show that there exists a positive integer N and $a_0, a_1, \ldots, a_N$ in F not all zero such that $a_N T^N + a_{N-1} T^{N-1} + \cdots + a_1 T + a_0 1_V$ is the zero linear operator, i.e., if $f = a_N t^N + a_{N-1} t^{N-1} + \cdots + a_1 t + a_0$, then $f(T) = 0$.

(b) Show that there is a unique monic polynomial $q \in F[t]$ satisfyingl $q(T) = 0$ and if $g \in F[t]$ is any polynomial satisfying $g(T) = 0$, then q divides g in $F[t]$, i.e., $g = qh$ for some $h \in F[t]$. (You may use the Division Algorithm without proving it.)

Problem 5. Let F be a field and $A \in \mathbb{M}_n(F)$.

(a) Suppose that A is an upper triangular matrix (i.e., $A_{ij} = 0$ for all $i < j$) with $A_{ii} = 0$, for $i = 1, \dots, n$. Prove that $A^n = 0$.

(b) Suppose that $F = \mathbb{C}$ and $\lambda_1, \dots, \lambda_k$ are all the distinct roots of the characteristic polynomial $f_A \in F[t]$ of A. Show that A is similar to an upper triangular matrix with diagonal entries ordered with all the λ_1's coming first on the diagonal, followed by all the λ_2's along the diagonal, etc. (So the diagonal entries look like $\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_k, \dots, \lambda_k$.)

(c) Let $T : V \to V$ be a linear operator on a finite dimensional complex vector space. Prove that $f_T(T) = 0$.

(d) (Extra Credit) Let F be an arbitrary subfield of $\mathbb{C}$ and $T : V \to W$ a linear transformation of finite dimensional vector spaces over F. Prove that $q_T \mid f_T$ in $F[t]$, where q_T is the minimal polynomial of T defined in Problem 4.

Problem 6. Let $T : V \to W$ be a linear transformation of finite dimensional inner product spaces of the same dimension.

(a) Show the following are equivalent:

- (i) $\langle T(v_1), T(v_2)\rangle_W = \langle v_1, v_2\rangle_V$ for all $v_1, v_2 \in V$. We say that T **preserves the inner product**.
- (ii) T is an isomorphism of vector spaces preserving inner products. We say that T is an **isometry**.
- (iii) T takes every ON basis of V to an ON basis of W.
- (iv) T takes some ON basis of V to an ON basis of W.
- (v) $||Tv|| = ||v||$ for every $v \in V$.

(b) Show two finite dimensional inner product spaces over F are **isometric** (i.e., there exists an isometry between them) if and only if they have the same dimension.

Problem 7. Let V and W be finite dimensional inner product spaces with orthonormal bases $\mathcal{B} = \{v_1, ..., v_n\}$ and $\mathcal{C} = \{w_1, ..., w_m\}$, respectively. Let $T : V \to W$ and $S : W \to V$ be two linear transformations. Suppose that $A = [T]_{\mathcal{B},\mathcal{C}} \in F^{m\times n}$ and $B = [S]_{\mathcal{C},\mathcal{B}} \in F^{n\times m}$ and are viewed as linear transformations of inner product spaces $F^{n\times 1} \to F^{m\times 1}$ and $F^{m\times 1} \to F^{n\times 1}$, respectively via the dot product (with $\mathcal{S}_{n,1}$ and $\mathcal{S}_{m,1}$ the respective standard bases). Then show all of the following (where $(A^*)_{ij} = \overline{A}_{ji}$ for all i, j) :

(a) $\langle T(v_i), w_j\rangle_W = A_{ji} = Ae_i \cdot e_j$ and $\langle v_i, S(w_j)\rangle_V = (B^*)_{ji} = e_i \cdot Be_j$ for all $1 \le i \le n$ and $1 \le j \le m$.

(b) There exists a unique linear transformation $T^* : W \to V$ such that $[T^*]_{\mathcal{C},\mathcal{B}} = A^*$ and it satisfies

$$\langle T(v_i), w_j\rangle_W = \langle v_i, T^*(w_j)\rangle_V \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m.$$

(c) The linear transformation $T^* : W \to V$ in (b) satisfies

$$(*) \qquad \langle Tv, w\rangle_W = \langle v, T^*w\rangle_V \quad \text{for all } v \in V \text{ and } w \in W.$$

(d) T^* is the unique linear transformation satisfying $(*)$. T^* is called the **adjoint** of T.

(e) $T^{**} := (T^*)^* = T$.

Problem 8. Let $A \in \mathbb{M}_n(F)$ be a matrix whose columns form an ON basis for $F^{n\times 1}$ under the dot product. Show A is invertible and the inverse of A is A^*.

Problem 9. Under the conditions and notation of Problem 7, show both of the following:

(a) T^*T is hermitian.

(b) $\langle T^*Tv, v\rangle_V$ is a non-negative real number for all $v \in V$.

Problem 10. Let V be a finite dimension inner product space and W a subspace of V. We know that $V = W \perp W^\perp$, i.e., if $v \in V$, then there exist unique $w \in W$ and $w^\perp \in W^\perp$ satisfying $v = w + w^\perp$.

(a) Let $T : V \to V$ be the linear operator satisfying $v = w + w^\perp \mapsto w - w^\perp$. Prove that T is both hermitian and an isometry. (See Problem 6 for the definition of an isometry.)

(b) Let $T : V \to V$ be a linear operator that is both hermitian and an isometry. Show that there exists a subspace W of V such that $T(v) = w - w^\perp$ where $v = w + w^\perp$ with $w \in W$ and $w^\perp \in W^\perp$.