# ANSWER SHEET

NAME Richard Sun 904444918

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**Problem 1** (25 points) Give specific examples of each of the following four items (You do not need to justify):

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a. A composition of two linear transformations $T : V \to W$ and $S : W \to X$ of vector spaces over $F$ such that the composition $S \circ T : V \to X$ is an isomorphism but $T$ is not an isomorphism.

$\dim V = \dim X \quad \dim V \neq \dim W$

$V = \mathbb{R}^2 \quad W = \mathbb{R}^3 \quad X = \mathbb{R}^2$

$T : \mathbb{R}^2 \to \mathbb{R}^3$ by $(a,b) \mapsto (a,b,0)$

$S : \mathbb{R}^3 \to \mathbb{R}^2$ by $(a,b,c) \mapsto (a,b)$

$S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ by $(a,b) \mapsto (a,b)$

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b. A real two dimensional vector space $V$ and a real four dimensional vector space $W$, and two linear operators $T$ and $S$ satisfying each of the following:

($i$) $T : V \to V$ has no eigenvalues

$\det(A - \lambda I) = \lambda^2 + 1$

$A - \lambda I = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

standard basis

$T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(1,0) = (0,1)$

$T(0,1) = (-1,0)$

($ii$) $S : W \to W$ has no eigenvalues.

$\det(A - \lambda I) = (\lambda^2 + 1)(\lambda^2 + 1)$

$\begin{pmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -1 & -\lambda \end{pmatrix}$

$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

$T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(e_1) = e_2$

$T(e_2) = -e_1$

$T(e_3) = -e_4$

$T(e_4) = e_3$

c. An infinite dimensional real inner product space $V$ together with an orthonormal basis and a subspace $W$ not $V$ that is also infinite dimensional.

$V = \{ (a_1, a_2, a_3 \ldots) \}$

Orthonormal basis for $V$: $\{e_1, e_2, e_3, \ldots\}$ where $e_i = (0, \ldots, 0, \overset{i^{th}}{1}, 0, \ldots)$

Inner product?

$W = \{ (0, a_2, a_3, \ldots) \}$

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d. A linear transformation $T : V \to W$ between two five dimensional real inner product spaces $V$ and $W$ having no elements in common and satisfying $||Tv||_W = ||v||_V$ for every $v$ in $V$. [Here $|| - ||_X$ denotes the inner product on the inner product space $X$.]

isomorphism

$V = \mathbb{R}^5$ with $\langle v, w \rangle$ = dot product

$W = M_{1\times5}\mathbb{R}$ with $\langle A, B \rangle = tr(AB)$ need transpose

$T: V \to W$ by $(a_1, a_2, a_3, a_4, a_5) \mapsto \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$

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**Problem 2** (25 points) Do all of the following (there are two parts):

a. Accurately state the full content of the following (named) theorems that we have proven in class.

**Existence of Linear Transformation Theorem (UPVS)**

Let $V$ be a fdvs/$F$ with basis $\{v_1 \ldots v_n\}$, $W$ a vs/$F$, and $w_1 \ldots w_n \in W$ not necessarily distinct. Then $\exists!\ T: V \to W$ linear such that $Tv_i = w_i$ $i = 1 \ldots n$.

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**The Matrix Theory Theorem**

Let $V, W$ be fdvs/$F$ with $\dim V = n$ and $\dim W = m$ and bases $B, C$ respectively. Then $\varphi: L(V,W) \to F^{m \times n}$ by $T \mapsto [T]_{B,C}$ is an isomorphism. In particular, $\dim L(V,W) = mn$.

b. Give a consequence (e.g., corollary or example or application) of three of the theorems stated in (a). (You do not need to justify.)

**First Consequence and of which theorem**

Orthogonal Decomposition Theorem

Let $V$ be a ips/F, $S \subset V$ a subspace, $v \in V$, $v = v_s + s^\perp$ where $v_s \in S$ and $s^\perp \in S^\perp$.

Then $v_s$, the orthogonal projection of $v$ on $S$, is closer to $V$ than any other vector in $S$, ie $\|v - v_s\| \leq \|v - w\|$ $\forall w \in S$, and $\|v - v_s\| = \|v - w\|$ iff $w = v_s$.

ie. $d(v, S) = \|v - v_s\|$ $v_s$ unique (Approximation Theorem)

**Second Consequence and of which theorem**

Matrix Theory Theorem

Let $V, W, X$ be fdvs/F with bases $B, C, D$ respectively, $T: V \to W$ and $S: W \to X$ linear.

Then $[S \circ T]_{B,D} = [S]_{C,D} \cdot [T]_{B,C}$

matrix multiplication

**Third Consequence and of which theorem**

UPVS

Let $V, W$ be fdvs/F. Then $V$ is isomorphic to $W$ iff $\dim V = \dim W$.

(Classification of fdvs)

**Problem 3** (25 points) Let $T : \mathbf{R} \to \mathbf{R}$ be the rotation by the angle $\theta$ in the plane perpendicular to $(1,-1,1)$ in $\mathbf{R}^3$ in the counterclockwise direction (with $(1,-1,1)$ pointing up). Compute the matrix representation of $T$ in the standard basis. You do not have to multiply the matrices that occur, but you do have to evaluate the inverse of a matrix if it occurs.

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Great Job!

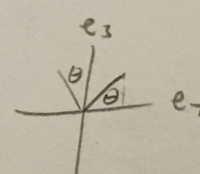
Orthonormalize basis

$u_1 = (1,-1,1)$ $\quad \|u_1\| = \sqrt{3}$

$\langle u_1, u_2 \rangle = 0$

$u_2 = (0,1,1)$ $\quad \|u_2\| = \sqrt{2}$

$u_3 = u_1 \times u_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (-1-1, -(1-0), 1-0) = (-2,-1,1)$ $\quad \|u_3\| = \sqrt{4+1+1} = \sqrt{6}$




$e_1 = \frac{1}{\sqrt{3}}(1,-1,1)$

$e_2 = \frac{1}{\sqrt{2}}(0,1,1)$ $\quad B = \{e_1, e_2, e_3\}$

$e_3 = \frac{1}{\sqrt{6}}(-2,-1,1)$

$[Te_1]_B = e_1$

$[Te_2]_B = \cos\theta\, e_2 + \sin\theta\, e_3$

$[Te_3]_B = \cos\theta\, e_3 - \sin\theta\, e_2$

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$[1_V]_{B,S} = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$[1_V]_{S,B} = [1_V]_{B,S}^{-1} = [1_V]_{B,S}^{T} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

$$[T]_S = [1_V]_{B,S}\,[T]_B\,[1_V]_{B,S}^{T}$$

**Problem 4** (25 points) Let $T : V \to V$ be linear operator of vector spaces over a field $F$. Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of $T$ and $v_1, \ldots, v_n$ are nonzero vectors of $V$ satisfying $T(v_i) = \lambda_i v_i$ for $i = 1, \ldots, n$. Show that $v_1, \ldots, v_n$ are linearly independent. [If you cannot do the general case, do the case $n = 2$ — it may even help you do the general case.]

$n=1$ $\{v_1\}$ is linearly independent as $v_1 \neq 0$

$n=2$ Let $\{v_1\}$ linearly independent.

$T(v_2) = \lambda_2 v_2$

suppose $\{v_1, v_2\}$ is linearly dependent

$Tv_2 = \lambda_2 v_2$

$Tv_2 - \lambda_2 v_2 = 0 = Tv_1 - \lambda_1 v_1$

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