Math 115AH Midterm 1

Rules: This is a **closed book exam**. You cannot use notes, books, the web, other people for help. You have until **2:00 Saturday 13 February 2021** to upload you answers into the appropriate boxes on gradescope. In the first box in gradescope you **must** write I followed all the rules, and then sign it.

Part I (20 points) In (a), (b), and (c) let V be a nonzero vector space over a field F (not necessarily finite dimensional) and S a nonempty subset of vectors in V. **Fully and accurately** define what it means for each of the following:

(a) A vector v in V to be in the **Span** of S.

(b) S to be a **linearly dependent** subset in V.

(c) S to be a **linearly independent** subset in V.

(d) A **matrix representation** of a linear transformation between vector spaces.

Part II (25 points) Give examples of each of the following (You do **not** need to justify):

(a) Let $\mathbb{M}_2(F)$ be the vector space over F of 2×2 matrices. Find a **basis** for the subspace of skew symmetric matrices $\{A \in \mathbb{M}_2(F) \mid A^t = -A\}$ in each of the following two cases:

(i) F is the field of two elements.

(ii) F is the field of three elements.

(b) Give a specific example of a vector space with 27 elements and subspace of it with 9 elements.

(c) Give an example of two vector spaces that are infinite dimensional (i.e., **not** finite dimensional) that are not isomorphic.

(d) Let $V = C[0,1]$, the real vector space of continuous functions $f : [0,1] \to \mathbb{R}$. Find an injective linear transformation (monomorphism) from the **real** vector space of complex matrices $\mathbb{M}_2(\mathbb{C})$ to V.

(e) Let V and W be **isomorphic** vector spaces. Give an example of a linear transformation $T : V \to W$ with T one-to-one but not onto and an example of a linear transformation $S : V \to W$ with S onto but not one to one.

Part III (40 points) Do all of the following:

(a) Fully and accurately **state and prove** the **Replacement Theorem**.

(b) Fully and accurately **state and prove** the **Isomorphism Theorem**.

Part IV (15 points) Let W be a finite dimensional vector space over F satisfying $W = W_1 + W_2 + W_3$ with each W_i a subspace of W, $i = 1, 2, 3$ and let

$$V = W_1 \times W_2 \times W_3 := \{(w_1, w_2, w_3) \mid w_i \in W_i,\ i = 1, 2, 3\},$$

a vector space with coordinate-wise operations. Prove that there exists a surjective linear transformation $T : V \to W$ that is an isomorphism if and only if $W_i \cap (W_j + W_k) = 0$ whenever i, j, k are distinct, $i, j, k \in \{1, 2, 3\}$.