ANSWER SHEET

NAME Richard Sun   904444918

Problem 1 (25 points) Give specific examples of each of the following four items (You do not need to justify):

a. Four 6-dimensional vector spaces V_1, V_2, V_3, V_4 over a field F containing no subspace of any of the others and a 3-dimensional subspace W_i of V_i for $i = 1, 2, 3, 4$.

$V_1 = \text{Span}(e_1, e_5, e_6, e_7, e_8, e_9)$ $W_1 = \text{Span}(e_1, e_5, e_6)$

2

$V_2 = \text{Span}(e_2, e_5, e_6, e_7, e_8, e_9)$ $W_2 = \text{Span}(e_2, e_5, e_6)$

if view Span(e_1, e_3, e_6) no [illegible] [illegible] e_6)
then span(e_2 [illegible], e_6) $\subset$ Span(e_2 [illegible] e_8)

$V_3 = \text{Span}(e_3, e_5, e_6, e_7, e_8, e_9)$ $W_3 = \text{Span}(e_3, e_5, e_6)$

$V_4 = \text{Span}(e_4, e_5, e_6, e_7, e_8, e_9)$ $W_4 = \text{Span}(e_4, e_5, e_6)$

b. An infinite dimension vector space V over a field F, an infinite dimensional subspace X of V but not V and three 3-dimensional subspaces W_1, W_2, and W_3 of V such that $W_1 + W_2 + W_3$ is 9-dimensional.

$V = P(F)$

$X = \{\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \dots \mid \alpha_1, \alpha_2, \alpha_3, \dots \in F\}$

$W_1 = (\alpha_0 + \alpha_1 x + \alpha_2 x^2 \mid \alpha_0, \alpha_1, \alpha_2 \in F\}$

5

$W_2 = \{\alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 \mid \alpha_3, \alpha_4, \alpha_5 \in F\}$

$W_3 = \{\alpha_6 x^6 + \alpha_7 x^7 + \alpha_8 x^8 \mid \alpha_6, \alpha_7, \alpha_8 \in F\}$

c. Two continuous real-valued functions on the unit interval $[0, 1]$ that are not polynomial functions but are linearly independent.

e^t, e^{2t}

6

d. A linear transformation $T : C^1(0, 1) \to C^3(0, 1)$ where $C^n(0, 1)$ is the vector space of real valued functions $f : [0, 1] \to \mathbf{R}$ that have continuous derivatives of order n on the closed interval $[0, 1]$.

$T: C'(0,1) \to C^3(0,1)$ by $f \mapsto 0$

6

Problem 2 (25 points) Do all of the following (there are two parts):

a. Accurately state the full content of the following (named) theorems that we have proven in class.

Toss In Theorem

Let V be a vs/F, and $S \subset V$ a linearly independent subset of V such that $V \setminus \mathrm{Span}\, S \neq \emptyset$. Let $v \in V \setminus \mathrm{Span}\, S$. Then $S \cup \{v\}$ is linearly independent.

Toss Out Theorem

Let V be a vs/F, and $V = \mathrm{Span}(v_1, \ldots, v_n)$ for some $v_1 \ldots v_n \in V$ for some n, Then a subset of $\{v_1, \ldots, v_n\}$ is a basis for V. In particular, V is fd

Replacement Theorem

Let V be a vs/F with $B = \{v_1, \ldots, v_n\}$ a basis for V. Let $0 \neq v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ for some $\alpha_1 \ldots \alpha_n \in F$ with $\alpha_i \neq 0$. Then $\{v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots v_n\}$ is a basis for V.

Dimension Theorem

Let V be a fdvs/F, and W a vs/F. Let $T: V \to W$ be a linear transformation, Then $\ker T$ and $\mathrm{Im}\, T$ are fdvs/F. $\dim V = \dim \ker T + \dim \mathrm{Im}\, T$

22

b. Give a consequence (e.g., corollary or example or application) of three of the theorems stated in (a). (You do not need to justify.)

First Consequence and of which theorem

Toss out Theorem.

Let V be a vs/F, and $V = Span(v_1, \ldots, v_n)$, Then V is finite dimensional.

Part of the theorem

Second Consequence and of which theorem

Dimension Theorem.

Let V, W be fdvs/F, and $T: V \to W$ linear. If $\dim W > \dim V$, then T cannot be onto.

Third Consequence and of which theorem

Toss In Theorem.

Let V be a fdvs/F. Then any linearly independent subset of V can be extended to a basis for V. (Extension Theorem)

Problem 3 (25 points) Define two linear transformations as follows:

$$T : \mathbf{R}^3 \to \mathbf{R}^4 \quad \text{is given by} \quad T(a, b, c) = (a + b,\ a - b, a, c)$$

and

$$S : \mathbf{R}^4 \to \mathbf{R}[t] \quad \text{is given by} \quad S(a, b, c, d) = (a + b) + ct + dt^3.$$

Find the dimensions of the kernel (null space) and the image (range) of all of the following: S, T, $S \circ T$. Label them carefully and give brief justifications (if you need more room continue on the back or another sheet).

The nullity $\dim(\ker(T)) = 0$

Reason

$T(a,b,c) = (a+b,\ a-b,\ a, c) = (0,0,0,0)$

$a = 0$

$c = 0$

$a+b = 0 \quad b = 0$

$a - b = 0$

$a, b, c = 0$, so only $(0,0,0)$ maps to $(0,0,0,0)$

The rank $\dim(\text{im}(T)) = 3$

Reason

$\dim \mathbb{R}^3 = \dim(\ker(T)) + \dim(\text{Im}(T))$ by the Dimension Theorem.

$3 = 0 + \dim(\text{Im}(T))$

$\dim(\text{im}(T)) = 3$

The nullity $\dim(\ker(S)) = 1$

15

Reason

$S(a,b,c,d) = (a+b) + ct + dt^3 = 0$

$a + b = 0 \quad a = -b$

$c = 0$

$d = 0$

There is one degree of freedom, so $\dim(\ker(S)) = 1$

The rank $\dim(\text{im}(S)) = 3$

Reason

$\dim \mathbb{R}^4 = \dim(\ker(S)) + \dim(\text{im}(S))$ by Dimension Theorem.

$4 = 1 + \dim(\text{im}(S))$

$\dim(\text{im}(S)) = 3$

The nullity dim $(\ker(S \circ T)) = 1$

Reason

$S \circ T(a,b,c) = 2a + at + ct^3 = 0$

$a = 0$

$c = 0$

There is one degree of freedom, so $\dim(\ker(T)) = 1$

The rank dim $(\operatorname{im}(S \circ T)) = 2$

Reason

$\{2+t, t^3\}$ is a basis for $\operatorname{im}(S \circ T)$

$S \circ T = S(T(a,b,c))$

$= S(a+b, a-b, a, c)$

$= ((a+b)+(a-b)) + at + ct^3$

$= 2a + at + ct^3$

Problem 4 (25 points) Prove the Toss In Theorem.

[Note there is a problem five which is to redo the test at home and turn it in next Monday.]

Let V be a vs/F, and $S \subset V$ a linearly independent subset of V such that $V \setminus \operatorname{Span} S \neq \emptyset$.

Let $v \in V \setminus \operatorname{Span} S$. v exists because $V \setminus \operatorname{Span} S \neq \emptyset$.

To show: $S \cup \{v\}$ is linearly independent

S not finite

Let $S = \{v_1, \ldots, v_n\}$. Suppose $S \cup \{v\}$ is linearly dependent. Then

$\alpha_0 v + \alpha_1 v_1 + \ldots + \alpha_n v_n = 0$ for some $\alpha_0 \ldots \alpha_n \in F$ not all 0.

Case 1: $\alpha_0 = 0$

Then $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$ $\alpha_1, \ldots \alpha_n$ not all 0. Then $S = \{v_1, \ldots v_n\}$ is linearly dependent, a contradiction.

Case 2: $\alpha_0 \neq 0$

Then α_0^{-1} exists.

$\alpha_0 v = -\alpha_1 v_1 - \ldots - \alpha_n v_n$

$v = -\alpha_0^{-1}\alpha_1 v_1 - \ldots - \alpha_0^{-1}\alpha_n v_n$.

So v is a linear combination of $\{v_1, \ldots v_n\}$, so $v \in \operatorname{Span} S$, contradicting $v \in V \setminus \operatorname{Span} S$. $\square$

23