Math 115AH Final

Rules: This is a **closed book exam**. You cannot use notes, books, the web, other people for help. You have until **8:00 AM PST Friday 19 March 2021** to upload your answers into the appropriate boxes on gradescope. In the first box in gradescope you **must** enter I followed all the rules, and then sign it.

1. State and prove the Counting Theorem.

2. Let V be a nonzero finite dimensional vector space over a field F and $\mathcal{B}$ a subset of V. Prove the following are equivalent:
   (a) $\mathcal{B}$ is a basis for V.
   (b) Among all spanning sets for V, the set $\mathcal{B}$ is a minimal one.
   (c) Among all linearly independent sets in V, the set $\mathcal{B}$ is a maximal one.

3. Let V and X be finite dimensional vector spaces over F and $T : V \to X$ a linear transformation. Prove both of the following:
   (a) If W is a subspace of V, then $V = \dim W + \dim W^0$.
   (b) $\dim \operatorname{im} T = \dim \operatorname{im} T^t$ where T^t is the transpose of T.
   [Recall that the *annihilator* of W is defined by $W^0 := \{f \in V^* \mid f(x) = 0 \text{ for all } x \in W\}$.]

4. State and prove the Universal Property of Vector Space (UPVS) (the Existence of Linear Transformations) without a finite dimensional assumption.
   [You can assume that all vector spaces have bases.]

5. Let V be a non-zero vector space over F (not necessarily finite dimensional), W_1, W_2 subspaces of V satisfying V is the direct sum $V = W_1 \oplus W_2$, and $P_i : V \to V$ the linear operator defined by $v \mapsto w_i$ in V if $v = w_1 + w_2$, with $w_i \in W_i$, for $i = 1, 2$.
   Let $T : V \to V$ be a linear operator. Prove that W_i, $i = 1, 2$, are both T-invariant if and only if $TP_i = P_iT$ for both $i = 1, 2$.
   [You may assume that for each $v \in V$, there exist unique $w_i \in W_i$, $i = 1, 2$, satisfying $v = w_1 + w_2$.]

6. Let V be a non-zero finite dimensional vector space over F and $T : V \to V$ a linear operator. Let $q_T \in F[t]$ be the minimal polynomial of T (the polynomial that you proved existed on Problem 4 of the Takehome). Prove that q_T and the characteristic polynomial f_T have the same roots in F.
   [You cannot use the Cayley-Hamilton Theorem.]

7. State and prove the Approximation Theorem.

8. Let V be a finite dimensional complex inner product space and $T : V \to V$ an arbitrary linear operator. Let $\lambda \in \mathbb{C}$. Show that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

9. State and prove Schur's Theorem.

10. Let $T : V \to V$ be an hermitian operator with V a finite dimensional real or complex inner product space. We call the hermitian operator T a *non-negative operator* if $\langle Tv, v\rangle \geq 0$ for all $v \in V$. Prove both of the following:
    (a) There exists a non-negative operator $S : V \to V$ satisfying $S^2 = T$, i.e., T has a square root.
    (b) The non-negative operator S in (a) is unique.