NAME Name Withheld
UPE-Certified Name Withholder™

Midterm I – Mathematics 115A

Friday, October 21, 2016

1	8
2	10
3	10
4	2+3+1
5	9
	43

1. Some quick questions – yes/no answers are fine.

(a) If $S = \{v_1, .., v_n\}$ **spans** a vector space of dimension n, does S have to be linearly independent?

No.

(b) Is the set of polynomials such that $p(1) = (18)^2 p(10) = p(5)$ a subspace of the vector space of polynomials?

Yes. – closed under addition & scalar multiplication.

(c) Is $\{4 + 4x + x^2, 1 + x, 5x - x^2, 1 + 3x^2\}$ linearly independent?

No. Too many vectors in P_2.

(d) If T is a linear transformation from a vector space V into V, is T^2 defined by $T^2(x) = T(T(x))$ also a linear transformation?

Yes.

(e) If T is a linear transformation from $\mathbb{R}^3$ to R^3, could it happen that $N(T)$ was a subspace of $R(T)$?

Yes.

8

2. (a) Assume that W_1 and W_2 are subspaces of a vector space V. Define $W_1 + W_2$ and state the relation of its dimension to the dimensions of W_1, W_2 and $W_1 \cap W_2$.

$$W_1 + W_2 = \{x : x = w_1 + w_2, \ w_1 \in W_1, \ w_2 \in W_2\}$$

Dimension Theorem:

$$\boxed{\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + \dim(W_1 \cap W_2).}$$

(b) Continuing part (a): Assume that $\dim(V) = n$, $\dim(W_1) = n - 2$ and $\dim(W_2) = n - 3$. What is the **smallest** possible value for $\dim(W_1 \cap W_2)$ when

(i) $\dim(V) = 4$

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$= (n-2) + (n-3) - \dim(W_1 \cap W_2)$$

$\dim(W_1 + W_2) \leq n$, so

$$\dim(W_1 \cap W_2) \geq (n-2) + (n-3) - n$$

$$\geq n - 5 = -1. \quad \text{This is impossible.}$$

$$\boxed{\dim(W_1 \cap W_2) \text{ can be } 0.}$$

(ii) $\dim(V) \geq 5$

By the previous analysis,

$$\dim(W_1 \cap W_2) \geq n - 5.$$

So $\boxed{\dim(W_1 \cap W_2) \text{ can be as small as } n-5.}$

16

3. Assume that T is a linear transformation of P_2 into P_3 and in terms of the ordered bases $\alpha = \{1, x, x^2\}$ and $\beta = \{1, x, x^2, x^3\}$, the matrix

$$[T]_\alpha^\beta = \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}.$$

What is $T(a + bx + cx^2)$?

$$T(1) = 1(1) + 0(x) + 0(x^2) + (-1)x^3 = 1 - x^3$$

$$T(x) = 0(1) + (-1)(x) + 1(x^2) + 1(x^3) = -x + x^2 + x^3$$

$$T(x^2) = 1(1) + 1(x) + 1(x^2) + 0(x^3) = 1 + x + x^2.$$

Thus $T(a + bx + cx^2)$

$$= [T]_\alpha^\beta \begin{bmatrix} a \\ b \\ c \end{bmatrix}_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + c \\ -b + c \\ b + c \\ -a + b \end{bmatrix}_\beta$$

$$= \boxed{(a + c) + (-b + c)x + (b + c)x^2 + (-a + b)x^3.}$$

10

4. Given that T is a linear transformation mapping the vector space V into the vector space W,

(a) Define the null space $N(T)$.

$N(T) = \{v \in V : T(v) = 0\}$ ✓

(b) Define the range space, $R(T)$, and prove that it is a subspace.

$\exists$, not $\forall$

$R(T) = \{w \in W : \forall v \in V \text{ s.t. } T(v) = w\}$

To show $R(T)$ is a subspace, we show closure under addition and scalar multiplication.

Suppose $w_1, w_2 \in R(T)$.

Then $w_1 = T(v_1)$, $w_2 = T(v_2)$.

By linearity, $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$, so $w_1 + w_2 \in R(T)$.

Suppose $w_1 \in R(T)$, $c \in F$.

Then $w_1 = T(v_1)$

By linearity, $cw_1 = cT(v_1) = T(cv_1)$, so $cw_1 \in R(T)$. ✓ ∎

(c) State (precisely) the theorem that relates the dimension of $R(T)$ to the dimension of $N(T)$, and prove it.

The Rank-Nullity Theorem states that

$$\boxed{\dim(N(T)) + \dim(R(T)) = \dim(V)}$$

Let $\alpha = \{v_1, \dots, v_k\}$ be a basis for $N(T)$. $\dim(N(T)) = k$.

Let $\beta = \{w_1, \dots, w_l\}$ be a basis for $R(T)$.

Consider a vector $x \in V$.

If $T(x) = 0$:

$x \in N(T)$ by definition. Then $x = a_1 v_1 + \dots + a_k v_k$, $a_i \neq 0$.
The space of choices for x is of dimension k.

$T(x) \in R(T)$ for all $x \in V$

If $T(x) \neq 0$:

$T(x) \in R(T)$ by definition. Then $T(x) = b_1 w_1 + \dots + b_l w_l$, $b_i \neq 0$
The space of choices for x is of dimension l.

No, set of x such that $Tx \neq 0$ is not a subspace

It is $V \cap (N(T))^c$

(S^c means the vectors not in S)

Therefore, $k + l = \dim(V)$, the space of choices for x,

and $\dim(N(T)) + \dim(R(T)) = \dim(V)$. ∎

5. Assume that $w \neq \underline{0}$ and that w belongs to the span of $S = \{v_1, .., v_p\}$ in the vector space V. Prove that you can replace one of the vectors in S by w without changing the span of S.

$w \in \text{span}(S)$.

So

$w = a_1 v_1 + \ldots + a_p v_p$ for some $a_1, \ldots, a_p \in F$.

Because $w \neq \underline{0}$, there exists some k s.t. $a_k \neq 0$. ✓

$w = a_1 v_1 + \ldots + a_k v_k + \ldots + a_p v_p$.

We solve for v_k:

$$-a_k v_k = -w + a_1 v_1 + \ldots + a_{k-1} v_{k-1} + a_{k+1} v_{k+1} + \ldots + a_p v_p.$$

$$v_k = \frac{1}{a_k} w + -\frac{1}{a_k}(a_1 v_1 + \ldots + a_{k-1} v_{k-1} + a_{k+1} v_{k+1} + \ldots + a_p v_p).$$ ✓

So v_k is a linear combination of $\{w, v_1, \ldots v_{k-1}, v_{k+1}, \ldots, v_p\}$

and by definition, $v_k \in \text{span}(\{w, v_1, \ldots, v_{k-1}, v_{k+1}, v_p\})$.

Consider an arbitrary vector $x \in \text{span}(S)$.

$x = b_1 v_1 + \ldots + b_k v_k + b_p v_p$. for some $b_1, \ldots, b_p \in F$.

Replace v_k with that expression:

$$x = b_1 v_1 + \ldots + b_k \left(\frac{1}{a_k} w + -\frac{1}{a_k}(a_1 v_1 + \ldots + a_{k-1} v_{k-1} + a_{k+1} v_{k+1} + \ldots + a_p v_p)\right) + \ldots + b_p v_p.$$

Simplify:

$$x = \frac{b_k}{a_k} w + \left(b_1 - \frac{b_k}{a_k} a_1\right) v_1 + \ldots + \left(b_{k-1} - \frac{b_k}{a_k} a_{k-1}\right) v_{k-1}$$
$$+ \left(b_{k+1} - \frac{b_k}{a_k} a_{k+1}\right) v_{k+1} + \ldots + \left(b_p - \frac{b_k}{a_k} a_p\right) v_p.$$

Then $x \in \text{span}(\{w, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\})$.

We can replace a vector in S by w without changing the span. ∎

This shows the span of S with v_k replaced by w contains the span of S. You should show the converse, too.