MATH 115A SECOND MIDTERM EXAMINATION

February 24h, 2017

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. You can not use any notes, books or electronic devices of any kind during the exam. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

I certify that the work appearing on this exam is completely my own

Signature: 王雪凝

Name: Xuening Wang

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10	10	10	10	10	50

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Problem 1. Let V, W, Z be vector spaces and $T : V \to W$ and $U : W \to Z$ be linear transformations.

(1) Suppose that $UT : V \to Z$ is one-to-one. Does it follow that T is one-to-one? If yes, give a proof. If not, provide a counter example (that is, give an example of vector spaces V, W, Z and linear transformations T, U such that UT is one-to-one but T is not one-to-one.

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Yes.

If $T(x) = T(y) \Rightarrow U(T(x)) = U(T(y))$ so $UT(x) = UT(y)$

And since UT is one to one. $\Rightarrow x = y$

Thus T is one to one.

(2) Suppose that $UT : V \to Z$ is one-to-one. Does it follows that U is one-to-one? If yes, give a proof. If not, construct a counter example. That is, give an example of vector spaces V, W, Z and linear transformations $T : V \to W$ and $U : W \to Z$ such that UT is one-to-one but U is not one-to-one.

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No.

Assume $T(x) = (x, 0)$ one to one.

$U(x, y) = (x, 0)$ Not one to one.

$UT(x) = U(x, 0) = (x, 0)$ is one to one.

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Problem 2. Let V, W be vector spaces, and let $T, U : V \to W$ be linear transformations. Let $\mathcal{L}(U, W)$ be the vector space of all linear transformations from V to W . Show that if $R(T) \cap R(U) = \{\vec{0}\}$, then $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(U, W)$.

Prove by contradiction

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If $\{T, U\}$ is linearly dependent

then $T = cU$ for some constant c

Let $w \in R(T)$ $w = T(v)$

then $w = T(v) = cU(v) = U(cv)$

Since $v \in V$, $cv \in V$.

so $w \in R(U)$ as well

$w \in R(T) \cap R(U)$

$R(T) \cap R(U) \neq 0$

$\Rightarrow \Leftarrow$

This proves the statement.

Problem 3. 1. Define the trace of an $n \times n$-matrix.

$$tr(A) = \sum_{i=1}^{n} A_{ii}$$

2. Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two $n \times n$ matrices A and B.

$$tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} B_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} B_{ki} A_{ik} = \sum_{k=1}^{n} (BA)_{kk}$$

$$= tr(BA)$$

3. Show that if two $n \times n$ matrices C and D are similar, then $\mathrm{tr}(C) = \mathrm{tr}(D)$.

Assume $C = Q^{-1} D Q$ for some Q

$$tr(C) = tr(Q^{-1} D Q)$$

$$= tr((Q^{-1} D) Q)$$

(using result from 2.)

$$= tr(Q(Q^{-1} D))$$

$$= tr(ID)$$

$$= tr(D)$$

Problem 4. Find the change of coordinates matrix that changes the basis β' into β in the vector space $P_2(\mathbb{R})$, where

$$\beta = \{x^2 - x,\ x^2 + 1,\ x - 1\}, \qquad \beta' = \{5x^2 - 2x - 3,\ -2x^2 + 5x + 5,\ 2x^2 - x - 3\}.$$

Let β_0 be the standard basis.

$\beta_0 = \{x^2, x, 1\}$

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad Q' = \begin{pmatrix} 5 & -2 & 2 \\ -2 & 5 & -1 \\ -3 & 5 & -3 \end{pmatrix}$$

Q changes basis from β to β_0, Q' changes basis from β' to β_0

Let S be the matrix that changes β' to β

$$S = Q^{-1}Q' \qquad S = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 5 & -2 & 2 \\ -2 & 5 & -1 \\ -3 & 5 & -3 \end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \qquad = \frac{1}{2}\begin{pmatrix} 10 & -12 & 6 \\ 0 & 8 & -2 \\ 6 & -2 & 4 \end{pmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \qquad = \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Problem 5. True/False

Mark every statement as either True (T) or False (F)

F (1) For any two $n \times n$ matrices A and B, we have

$$\operatorname{rank}(A) + \operatorname{rank}(B) = \operatorname{rank}(A + B)$$

T (2) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ such that $\operatorname{nullity}(T) = 2$.

F (3) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear transformation, then $\mathbb{R}^2 = N(T) \oplus R(T)$.

F (4) If $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent, then each vector in S is a linear combination of other vectors in S.

F (5) For any $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ and $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^3$ there is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T(\vec{v}_1) = \vec{w}_1, \qquad T(\vec{v}_2) = \vec{w}_2.$$

T (6) If $A, B \in M_{n \times n}(\mathbb{R})$ are such that $AS = SB$ for an invertible matrix $S \in M_{n \times n}(\mathbb{R})$, then A and B have the same eigenvalues.

F (7) Every linear transformation of the plane has at least one real eigenvalue. $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

F (8) If you multiply every entry of a square matrix by a constant, the determinant of the matrix is multiplied by the same constant.

T (9) $\det(AB) = \det(A) \cdot \det(B)$ for any two square matrices A and B.

F (10) $\det(A^t) = \frac{1}{\det(A)}$.

$N(T) \cap R(T)$

$T(x) = 0$

x in $R(T)$?

$T(y) = x$

$T^2(y) = 0$. but $T(y) \neq 0$

10

$T(x) = x - 5$

$T(5) = 0$

$T(10) = 5$