Math 115A
Midterm 2
2016 November 14

Name: Cetorelli, Ty
Please put your last name first and print clearly

UID: 904-868-337

Signature:

– NO BOOK – NO CALCULATOR –

You can use both sides of the sheet for each problem. If you use extra sheets, write your name on them, and indicate which problem you are treating.

All answers must be stated clearly.

All answers must be justified (except for Problem 1).

Verbiage will be ignored or counted negatively if wrong. Good luck!

– SCORE –

1. 1 2. 0

3. 0 4. 5

5. 3

Total 9

Problem 1. [5 pts]

Answer only by "True" (T) or "False" (F) **without justification.**

(a) Two finite-dimensional vector spaces with the same dimension are always isomorphic.

False

(b) The sum $S + T$ of two linear isomorphisms $S, T : V \to W$ is again a linear isomorphism.

True

(c) Consider a square matrix of the form $A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$, with B a square matrix. Then A is invertible if and only B and D are invertible.

True

(d) A square matrix whose characteristic polynomial splits completely is diagonalizable.

True

(e) For $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ to be diagonalizable, it suffices that the geometric multiplicity of every eigenvalue of A equals its algebraic multiplicity.

True

1/5

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Problem 2. [5 pts]

Suppose that $T : V \to V$ is diagonalizable, with $\dim(V) = n$. Let $f_T(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be its characteristic polynomial. Prove that $f_T(T)$ is the zero linear transformation, where we recall that $f_T(T) = (-1)^n T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 \operatorname{Id}_V$. [Note: This is true without assuming T diagonalizable, by the Cayley-Hamilton Theorem, but you cannot invoke that theorem here, of course.]

~~Characteristic polynomial is not dependent on~~

$f_T(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$

T is diagonalizable so $f_T(t)$ splits completely

Say $f_T(t)$ has r terms

Then

0/5

Problem 3. [4 pts]

Give an **example** of a linear transformation $T : V \to V$ with two eigenvalues: $\lambda_1 = 4$ with algebraic multiplicity 3 and geometric multiplicity 2, and $\lambda_2 = \sqrt{3}$ with algebraic multiplicity 4 and geometric multiplicity 3.

$f_A(t) = (t-4)^3 (t-\sqrt{3})^4$ So $m_1 = 3$ and $m_2 = 4$

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} \end{bmatrix}$$

2 vectors from E_{λ_1} in this matrix A implies $\dim(E_{\lambda_1}) = 2$, the geometric multiplicity of λ_1

3 vectors from E_{λ_2} in this matrix A implies $\dim(E_{\lambda_3}) = 3$, the geometric multiplicity of λ_2

0/4

Problem 4. [5 pts]

Consider the linear transformation $T = L_A : \mathbb{R}^2 \to \mathbb{R}^3$ given by the matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ 4 & 2 \end{pmatrix}$. Give the matrix of T with respect to the ordered bases

$$\mathcal{B} = \left\{ \underset{B_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{B_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right\} \text{ and } \mathcal{C} = \left\{ \underset{c_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{c_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{c_3}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} \right\}.$$

Give $[T]_\beta^{\mathcal{C}}$

$$T(B_1) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$T(B_2) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = a_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} a_2 + b_2 + c_2 \\ a_2 + c_2 \\ a_2 + b_2 + 2c_2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & 5 & 4 \\ 1 & 0 & 1 & 3 & 1 \\ 1 & 1 & 2 & 8 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 5 & 4 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$$

so $a_1 = 0$, $b_1 = 2$, $c_1 = 3$ $\Rightarrow [T(B_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

and $a_1 = -1$, $b_1 = 3$, $c_1 = 2$ $\Rightarrow [T(B_2)]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

and Thus $\boxed{[T]_\beta^{\mathcal{C}} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}}$ ✓

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Problem 5. [6 pts]

Prove that the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3\,x_1 - 2\,x_3 \\ 2\,x_1 + x_2 - 2\,x_3 \\ 4\,x_1 - 3\,x_3 \end{bmatrix}$$

is diagonalizable, and diagonalize it.

3/6

Let A be the matrix of the linear transformation

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 4 & 0 & -3 \end{bmatrix} \quad \text{Let } \alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$A = [T]_\alpha^\alpha$ obviously,

$$f_A(t) = |A - tI_3| = \begin{vmatrix} 3-t & 0 & -2 \\ 2 & 1-t & -2 \\ 4 & 0 & -3-t \end{vmatrix}$$

$$= (3-t)\begin{vmatrix} 1-t & -2 \\ 0 & -3-t \end{vmatrix} - 0 + (-2)\begin{vmatrix} 2 & 1-t \\ 4 & 0 \end{vmatrix}$$

$$= (3-t)(-3-t+3t+t^2-0) + (-2)(0-4+4t)$$

$$= (3-t)(t^2+2t-3) - 8t + 8$$

$$= 3t^2+6t-9-t^3-2t^2+3t-8t+8$$

$$= -t^3+t^2+t-1 = -t^2(t-1)+1(t-1)$$

$$= (-t^2+1)(t-1) = -(t^2-1)(t-1)$$

$$f_A(t) = -(t+1)(t-1)^2$$

Since $f_A(t)$ is completely split, A is diagonalizable and, thus, T is too. ~~(This is a theorem)~~

NOPE

$$\begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 4 & 0 & -3 \end{bmatrix}$$

$$f_A(t) = -(t+1)(t-1)^2$$

So $\lambda_1 = -1$, (with $m_1 = 1$) and
$\lambda_2 = 1$ (with $m_2 = 2$)

E'vector for λ_1:

$$A - \lambda_1 I_3 = \begin{bmatrix} 3+1 & 0 & -2 \\ 2 & 1+1 & -2 \\ 4 & 0 & -3+1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 2 & -2 \\ 4 & 0 & -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & -2 & 0 \\ 2 & 2 & -2 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_1 - \tfrac{1}{2}x_3 &= 0 \\ x_2 &= 0 \end{aligned}\right\} \text{e'vector for } \lambda_1: \ v_1 = \cancel{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$$

E'vector for λ_2:

$$A - \lambda_2 I_3 = \begin{bmatrix} 3-1 & 0 & -2 \\ 2 & 1-1 & -2 \\ 4 & 0 & -3-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 4 & 0 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 4 & 0 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_1 - x_3 &= 0 \\ x_2 &\ \text{free} \end{aligned}\right\}$$ e'vectors for λ_2 (that are linearly <u>in</u>dependent): $V_{2,1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $V_{2,2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let eigenbasis $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

So $[Id]^{\alpha}_{\beta} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Find $\left([Id]^{\alpha}_{\beta}\right)^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left([Id]^{\alpha}_{\beta}\right)^{-1} = [Id]^{\beta}_{\alpha} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[Id]^{\beta}_{\alpha} A [Id]^{\alpha}_{\beta}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ -4 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ & & \\ & & \end{bmatrix}$$

$$\cancel{= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} 9 & 1 & -7 \\ 4 & 0 & -3 \\ 12 & 1 & -9 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} & & \\ & & \end{bmatrix}}$$