Final

3/14/2020

Math115A
Nadja Hempel
nadja@math.ucla.edu

Name:

UID:

Problem	Points	Score
1	24	
2	6	
3	4	
4	10	
5	5	
6	17	
7	4	
8	5	
9	10	
10	5	
11	10	
Total	100	

Exercise 1. (24pt) Let $T : \mathcal{P}_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, $ax^3 + bx^2 + cx + d \mapsto \begin{pmatrix} a+d & b+c \\ b+c & a+d+b+c \end{pmatrix}$

Let $\mathcal{B} = \left(1, x, x^2, x^3\right)$, $\mathcal{B}' = \left(1, 1+x, x+x^2, x^2+x^3\right)$,

$$\mathcal{C} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

be the standard basis for $M_{2\times 2}(\mathbb{R})$ and

$$\mathcal{C}' = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Given: $\mathcal{C}'$ is a basis for $M_{2\times 2}(\mathbb{R})$.

(1) Show that T is linear. (2pt)
(2) Show that $\mathcal{B}'$ is a basis. (3pt)
(3) Give a basis for $\ker(T)$ and $\operatorname{im}(T)$. (No need to justify) (4pt)
(4) Is T is isomorphism?(1pt) Justify your answer. (1pt)
(5) Give $[T]^{\mathcal{C}}_{\mathcal{B}}$ (2pt), $[1_{\mathcal{P}_3(\mathbb{R})}]^{\mathcal{B}}_{\mathcal{B}'}$ (2pt) and $[1_{M_{2\times 2}(\mathbb{R})}]^{\mathcal{C}'}_{\mathcal{C}}$ (3pt). (7pt)
(6) Use (5) to compute $[T]^{\mathcal{C}'}_{\mathcal{B}'}$. (2pt)
(7) Compute $[3x^3 + 2x^2 - x + 1]_{\mathcal{B}'}$ and $[T(3x^3 + 2x^2 - x + 1)]_{\mathcal{C}'}$. (4pt)

Exercise 2. (6pt) Consider the matrix

$$A = \begin{pmatrix} 0 & c & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where c is a real number.

(1) (3pt) Find the eigenvalues of $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ in terms of c.
(2) (3pt) For which values of c is $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ diagonalizable? Justify your answer.

Exercise 3. (4pt) Suppose V and W are vector spaces over $\mathbb{R}$ and that $T : V \to W$ is an injective linear transformation. Show that $(v_1, v_2, ..., v_n) \in V^n$ is linearly independent if and only if $(T(v_1), T(v_2), ..., T(v_n)) \in W^n$ is linearly independent.

Exercise 4. (10pt) Suppose V is a finite dimensional vector space over $\mathbb{R}$ and that $T : V \to V$ is a linear transformation with the property that $T^2 = T$.

(1) (2pt) Prove by induction on n that $T^n = T$ for all $n \in \mathbb{N}$.
(2) (4pt) Show that $V = \ker(T) \oplus \operatorname{im}(T)$.
(3) (2pt) Prove that for every $v \in \operatorname{im}(T)$, $T(v) = v$.
(4) (2pt) Show that if $\ker(T) = \{0\}$, then $T = 1_V$.

Exercise 5. (5pt) Let $T : V \to V$ and $S : V \to V$ be linear transformations. Suppose v is

- an eigenvector of T with corresponding eigenvalue λ.
- an eigenvector of S with corresponding eigenvalue μ.

(1) (2pt) Show that v is an eigenvector of $S \circ T$ corresponding to $\lambda\mu$.

Now let $G_{\mu\lambda}$ be the geometric multiplicity of $\mu\lambda$ for $S \circ T$. Moreover, let E_λ be the Eigenspace corresponding to λ with respect to T and E_μ be the Eigenspace corresponding to μ with respect to S.

(2) (3pt) Show that $G_{\mu\lambda} \geq \dim(E_\lambda \cap E_\mu)$.

Exercise 6. (17pt) Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $\mathbb{F}$ be the set of scalar, i. e. $\mathbb{R}$ or $\mathbb{C}$. Suppose $T : V \to V$ is an **injective** linear transformation. Define a new function $\langle -, - \rangle_T : V \times V \to \mathbb{F}$ by $\langle u, v \rangle_T = \langle T(u), T(v) \rangle$ for all $u, v \in V$.

(1) (5pt) Show that $\langle -, - \rangle_T$ is an inner product on V.
(2) (2pt) Give a counter example that $\langle -, - \rangle_T$ is **not** an inner product when T is **not** injective. Explain why it is a counter example.
Hint: Try with $V = \mathbb{R}^2$ and $T = T_A$ for some $A \in M_{2\times 2}(\mathbb{R})$
(3) (5pt) Let $V = \mathbb{R}^3$ with the standard inner product and

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 3 & 9 \\ 2 & 0 & 0 \end{pmatrix}.$$

Apply the Gram-Schmidt process to (e_1, e_2, e_3) with respect to $\langle -, - \rangle_{T_A}$

Attention: This is NOT The standard inner product.
(4) (2pt) Give an orthornormal basis for $(V, \langle -, - \rangle_{T_A})$ using part (3).
(5) Using the above, give a basis for $\{e_1\}^\perp$ in $(\mathbb{R}^3, \langle -, - \rangle_{T_A})$. (1pt) Justify that it is indeed a basis for $\{e_1\}^\perp$. (2pt)

Exercise 7. (4pt) Let V be a vector space of dimension $n \in \mathbb{N}$ and $V_1, \dots, V_n$ be subspaces of V of dimension $n-1$. Show by induction on k that if $1 \leq k \leq n$ then an intersection of k subspaces of dimension $n-1$ always has dimension at least $n-k$.

Exercise 8. (5pt) Let $(V, \langle , \rangle)$ be an inner product space and $T : V \to V$ be a linear transformation. Let $S_1 = T + T^*$ and $S_2 = T \circ T^*$. Show that

(1) (3pt) $S_1^* = S_1$.
(2) (2pt) $S_2^* = S_2$.

Exercise 9. (10pt) Answer the following questions and justify your answer.

(1) For what values of a, b, c, d is:
  (a) $\{(x, y, z) : x + y = a, b^2x + cy^2 + cdxy = 0\}$ a subspace of $\mathbb{R}^3$. (2pt)
  (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, define by $(x, y, z) \mapsto (ax + y, by^2, z + c)$ linear? (2pt)
  (c) $\{(a, b, 0), (a, c, 0), (1, 1, d)\}$ a basis for $\mathbb{R}^3$. (2pt)

(2) Choose λ, μ, ν in $\mathbb{R}$ such that (Show your work! No additional justification needed)
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $(x, y) \mapsto (\lambda x + \mu y, \nu(x + y), x + y)$ is not injective. (2pt)
   (b) $U : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $(x, y, z) \mapsto (\lambda x + \mu y + \nu z, x + z)$ is not surjective. (2pt)

Exercise 10. (5pt) Let V be a vector space over $\mathbb{R}$, $\lambda \in \mathbb{R}$ and $v \in V$. Assume that $\lambda v = 0_V$. Show that $\lambda = 0$ or $v = 0_V$

*(You should **ONLY** use the **axioms of a vector space and cancelation law**. You **CANNOT** use that $\mu 0_V = 0_V$ for all $\mu \in \mathbb{R}$. For each step state clearly which axiom or result you are using.)*

Exercise 11. (10pt) Please answer the following questions on the "T/F final" quiz on CCLE. ATTENTION: the time limit to enter your answers once you open the quiz is 1 hour.

Let U_1, U_2, and U_3 be finite dimensional vector spaces such that $\dim(U_1) = n_1$, $\dim(U_2) = n_2$, and $\dim(U_3) = n_3$. Moreover let $T_1 : U_1 \to U_2$ and $T_2 : U_2 \to U_3$ be linear transformations.

Let V be an inner product space, W be a subspace of V, $S_1 : V \to V$ and $S_2 : V \to V$ be linear transformations.

(1) Let $v \in V$ and $\lambda, \mu \in \mathbb{R}$ be such that $\lambda \cdot v = \mu \cdot v$. Then $\lambda = \mu$.
(2) $W \cup W^{\perp} = V$.
(3) Every subspace of $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$ or $\mathbb{R}$.
(4) If v is an eigenvector of S_1 with corresponding eigenvalue 3, and an eigenvector of S_2 with corresponding eigenvalue 2, then v is an eigenvector of $S_1 \circ S_2$ with corresponding eigenvalue 6.
(5) Let A be an $n \times n$ matrix with $n > 0$. If $A^m = 0$ for some $m \in \mathbb{N}$, then A is not invertible.
(6) If $(v_1, \ldots, v_n)$ is an orthogonal tuple in V then $(\frac{v_1}{\|v_1\|}, \ldots, \frac{v_n}{\|v_n\|})$ is orthonormal.
(7) Let $\mathcal{B}$ be an orthogonal basis for V and $v \in V$ be such that $[v]_{\mathcal{B}} = (1, 0, \ldots, 0)$. Then
$$||v|| = 1.$$
(8) Let A be an $n \times n$ matrix. If $B^{-1}AC$ is a diagonal matrix for some $n \times n$-matrix C and invertible $n \times n$-matrix B, then T_A is diagonalizable.
(9) If $T_2 \circ T_1$ is injective then $n_1 \leq n_2$.
(10) If $T_2 \circ T_1$ is surjective then $n_2 \geq n_3$.