Final
Math 115A Section 5
Winter 2021

Directions: This is an **open book and open note exam**. You must give proper justification for all of your answers, citing the formal definitions as appropriate. You need to prove everything you claim, unless you are citing a result from lecture or homework.

Upload your solutions on to gradescope by **Wednesday, March 17th at 8 AM PST**. Give yourself at least 30 minutes to do the upload, just so that there is no risk of missing the deadline.

There are 10 problems on this exam, each worth 10 points.

Good luck!

Problem 1: (10 points total) Let V be a finite dimensional vector space over $\mathbb{F}$, and let $T \in \mathcal{L}(V)$. Recall for $\lambda \in \mathbb{F}$, we define $E_\lambda = \{v \in V : T(v) = \lambda v\}$.

a) (1 point) Let $\lambda \in \mathbb{F}^\times$. Prove $E_\lambda \subseteq \text{im}(T)$.

b) (3 points) Suppose T is diagonalizable. Prove that if $E_0 \subseteq \text{im}(T)$, then $\text{im}(T) = V$, $E_0 = 0$, and T is invertible.
Hint: If $E_0 \subseteq \text{im}(T)$, then, with (a), we get $E_\lambda \subseteq \text{im}(T)$ for all $\lambda \in \mathbb{F}$.

c) (3 points) Let $\lambda \in \mathbb{F}^\times$. Suppose $\text{im}(T) \subseteq E_\lambda$. Prove T is diagonalizable.

d) (3 points) Suppose $\text{im}(T) \subseteq E_0$. Prove that if T is diagonalizable, then $T = 0$.

Remark: The converses to (b) and (d) are trivially true.

Problem 2: (10 points total) Let V be a finite dimensional vector space over $\mathbb{F}$ and let $T \in \mathcal{L}(V)$ be such that $\text{rank}(T) = 1$ and $T^2 \neq 0$.

a) (3 points) Prove that every nonzero vector in $\text{im}(T)$ is an eigenvector of T.

b) (3 points) Prove T has a nonzero eigenvalue.

c) (4 points) Prove T is diagonalizable.

Problem 3: (10 points total) Let V be a finite dimensional vector space over $\mathbb{R}$ and let $T \in \mathcal{L}(V)$ be such that $T^2 = 5T - 6 \cdot Id_V$.

a) (3 points) Find, with proof, all possible eigenvalues of T. $\boxed{\text{Box}}$ in your final answer.

b) (3 points) Prove T is invertible.

c) (4 points) Prove T is diagonalizable.

Problem 4: (10 points total plus 3 bonus points) Take $V = M_n(\mathbb{F})$ as a vector space over $\mathbb{F}$. For $A \in M_n(\mathbb{F})$, we may define the linear map $m_A : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ given by $m_A(B) = AB$.

a) (3 points) Suppose A is invertible. Prove that if A is an eigenvector of m_A, then A is a scalar multiple of the identity matrix $I_n \in M_n(\mathbb{F})$.

b) (3 points) Let $n = 2$. Find, with proof, a nonzero matrix $A \in M_2(\mathbb{F})$ such that $A \in \ker(m_A)$. $\boxed{\text{Box}}$ in your final answer. (Your example should work over an arbitrary field $\mathbb{F}$).

c) (4 points) Let $n = 2$. Take the ordered basis $\beta = \{E_{11}, E_{21}, E_{12}, E_{22}\}$ of $M_2(\mathbb{F})$ (pay close attention to the order). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for $a, b, c, d \in \mathbb{F}$. Find $[m_A]_\beta$. Show your work. $\boxed{\text{Box}}$ in your final answer.

d) (Bonus: 3 points) For arbitrary n, give an ordered basis β of $M_n(\mathbb{F})$ for which $[m_A]_\beta$ is block-diagonal. Explain why your chosen basis will work (though you do not need to give a formal proof). Then give a formula for $\text{rank}(m_A)$ in terms of $\text{rank}(A)$. Justify your claim (though you do not need to give a formal proof). $\boxed{\text{Box}}$ in your final answers.

Problem 5: (10 points total) Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Define V^* as the vector space $\mathcal{L}(V, \mathbb{F})$.

a) (3 points) For any $v \in V$, define $f_v : V \to \mathbb{F}$ via $f_v(x) = \langle x, v \rangle$. Prove $f_v \in V^*$. That is, prove f_v is linear.

b) (3 points) Prove that if $f_v = f_w$, then $v = w$.

c) (4 points) Let $v_1, ..., v_n$ be an orthonormal basis of V. Let $h \in V^*$. Prove $h = \sum_{i=1}^{n} a_i f_{v_i}$ for some scalars $a_i \in \mathbb{F}$.

Problem 6: (10 points total) Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

a) (3 points) Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove $\ker(T) = \operatorname{im}(T)^{\perp}$.
(Do not simply cite a homework problem here).

b) (4 points) Let $S \in \mathcal{L}(V)$ be arbitrary. Prove $\ker(S^* \circ S) = \ker(S)$.

c) (3 points) Let $R \in \mathcal{L}(V)$ be arbitrary. Prove $\ker((R^* \circ R)^2) = \operatorname{im}(R^* \circ R)^{\perp}$.

Problem 7: (10 points total) Let V be a finite dimensional vector space over $\mathbb{R}$, and let $T \in \mathcal{L}(V)$. Suppose we have that the characteristic polynomial of T is $c_T(x) = x^2(x-2)(x+1)(x^2+1)$.

a) (3 points) Is T invertible? Justify your claim.

b) (3 points) Find a $c \in \mathbb{R}$ such that $c \neq 0$ and $T + c \cdot Id_V$ is not invertible. Justify your claim. $\boxed{\text{Box}}$ in your final answer.

c) (4 points) Prove $T^2 + 2T + Id_V \neq 0$.

Problem 8: (10 points total) Let $A \in M_6(\mathbb{C})$. Suppose we have that the characteristic polynomial of A is $c_A(x) = x^2(x-2)(x+1)(x^2+1)$.

a) (2 points) Find all eigenvalues of A, as well as their algebraic multiplicities. For each eigenvalue, list all possible geometric multiplicities. $\boxed{\text{Box}}$ in your final answers.

b) (3 points) Find, with proof, a matrix $D \in M_6(\mathbb{R})$ which is diagonalizable over $\mathbb{C}$ and has the same characteristic polynomial as A. $\boxed{\text{Box}}$ in your final answer.

c) (2 points) Find, with proof, an example of a matrix $B \in M_2(\mathbb{C})$ with $c_B(x) = x^2$ and B not diagonalizable over $\mathbb{C}$. $\boxed{\text{Box}}$ in your final answer.

d) (3 points) Find, with proof, an example of a matrix $A \in M_6(\mathbb{C})$ with $c_A(x) = x^2(x-2)(x+1)(x^2+1)$ and A not diagonalizable over $\mathbb{C}$. $\boxed{\text{Box}}$ in your final answer.

Remark: Notice that the matrices in *b* and *d* have the same characteristic polynomial, but are not similar.

Problem 9: (10 points total) Give the real vector space $V = M_2(\mathbb{R})$ its standard inner product via $\langle A, B \rangle = tr(AB^t)$.

a) (5 points) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 12 & 3 \\ 1 & 1 \end{bmatrix}$. Let $W = \text{span}(\{A, B, C\})$. Find an orthonormal basis of W. Show your work. $\boxed{\text{Box}}$ in your final answer.

b) (3 points) Let $X = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$. Find the matrix $Y \in W$ with $||X - Y||$ as small as possible. Justify your answer. $\boxed{\text{Box}}$ in your final answer.

c) (2 points) Find, with proof, an orthonormal basis of $W^\perp$. $\boxed{\text{Box}}$ in your final answer.

Problem 10: (10 points total) Endow $V = \mathbb{C}^4$ with the standard inner product. Let $T \in \mathcal{L}(V)$ be given by

$$T((a, b, c, d)) = (-b, a, -2d, 2c)$$

for any $(a, b, c, d) \in \mathbb{C}^4$.

a) (3 points) Let $\beta = \{e_1, e_2, e_3, e_4\}$. Find $[T]_\beta$ and $[T^*]_\beta$. Show your work. $\boxed{\text{Box}}$ in your final answer.

b) (1 point) Prove T^* is a scalar multiple of T and that $TT^* = T^*T$.

c) (2 points) Find all eigenvalues of T, along with their algebraic and geometric multiplicities. Show your work. $\boxed{\text{Box}}$ in your final answer.

d) (4 points) Find an orthonormal basis γ of V with $[T]_\gamma$ diagonal. Show your work. $\boxed{\text{Box}}$ in your final answer.