Math 115A
Linear Algebra

Quiz 1

Instructions: You have 60 minutes to complete the exam. There are five problems worth a total of 38 points. You may not use any books or notes. Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not forget to write your name and UID in the space below.

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Question	Points	Score
1	6	2
2	12	3
3	5	5
4	7	2
5	8	6
Total:	38	18

Problem 1

Recall that $\mathbb{R}^1 = \{ (x) : x \in \mathbb{R} \}$ is a vector space over $\mathbb{R}$ with coordinatewise addition and scalat multiplication.

2 (a) [2pts.] List every subspace of $\mathbb{R}^1$. You don't need to prove your claim in this part.

0 (b) [4pts.] Suppose V is a subspace of $\mathbb{R}^1$.

Prove that V has to be equal to one of the subspaces you wrote down in part (a).

a. $\{0\}$, $\mathbb{R}^1$

b. Let V be as in the question statement above. We want to show that V must equal $\mathbb{R}^1$ or $\{0\}$ ~~to be a subspace~~. The conditions of a subspace are as follows

1. $0 \in V$
2. If $v_1 \in V$, $w_1 \in V$, then $v_1 + w_1 \in V$
3. If $v \in V$, $\lambda \in \mathbb{R}$, then $\lambda v \in V$.

NO

Suppose V is an arbi~~trary set~~. To fulfill condition 3, V must span ~~the whole real line~~ $\mathbb{R}^1$, as any number multiplied with a scalar must be positive or negative, and can range from $-\infty$ to $+\infty$. Furthermore, for the set $\mathbb{R}^1$, condition 2 holds true, as any $r_1 \in \mathbb{R}^1$ added with another $r_2 \in \mathbb{R}^1$ is still in $\mathbb{R}^1$. Finally, $0 \in \mathbb{R}^1$, so condition 1 holds.

In the case of the $\{0\}$, the only element is 0. $0 \in 0$, so condition 1 holds. $0 + 0 = 0$, so condition 2 holds. $\lambda \in \mathbb{R}$, $0\lambda = 0$, as we know from real numbers, so condition 3 holds.

Therefore only $\mathbb{R}^1$ and $\{0\}$ fulfill the three conditions of subspace.

not what you need to prove

Problem 2.

Suppose X is a nonempty set.

In class and on the homework, we proved that the set of real-valued functions from X,

$$\mathcal{F} = \{f : X \to \mathbb{R}\}$$

is a vector space over $\mathbb{R}$ when equipped with pointwise addition and scalar multiplication.

0 (a) [2pts.] Give the definition of addition and scalar multiplication in $\mathcal{F}$.

0 (b) [2pts.] Verify the eighth axiom of a vector space for $\mathcal{F}$, that is, that for all $\lambda, \mu \in \mathbb{R}$, and for all $f \in \mathcal{F}$, $(\lambda + \mu)f = \lambda f + \mu f$.

0 (c) [3pts.] Fix $x_0, x_1 \in X$. Prove that $T : \mathcal{F} \to \mathbb{R}$, $f \mapsto f(x_0) + f(x_1)$ is linear.

Now take $X = \mathbb{R}$ so that $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$.

2 (d) [2pts.] Is the subset $\{f \in \mathcal{F} :\ f(0) = 1\}$ a subspace of $\mathcal{F}$?
Prove your claim.

1 (e) [3pts.] Is the subset $\{f \in \mathcal{F} :\ f(-1) \cdot f(1) = 0\}$ a subspace of $\mathcal{F}$?
Prove your claim.

a. Addition: $f_1, f_2 \in F$, then $(f_1) + (f_2) = (f_1 + f_2)$. ?
Multiplication: $f_1 \in F$, $\lambda \in \mathbb{R}$, then $\lambda(f_1) = (\lambda \cdot f_1)$

b. We set a variable? $\alpha = (\lambda + \mu)$. Then $(\lambda + \mu)f = \alpha(f)$.
Using the definition of scalar multiplication, we know $\alpha(f) = (\alpha \cdot f)$. not real numbers
Substitute $\lambda + \mu$ back and we get $((\lambda + \mu) \cdot f)$. Then we know distribution of real numbers, we can rewrite $((\lambda + \mu) \cdot f) = (\lambda \cdot f + \mu \cdot f)$.
This is the righthand side of the original equation. Therefore Axiom 8 holds.

c. Consider the problem statement. We want to show $T : F \to \mathbb{R}$, $f \to f(x_0) + f(x_1)$ is linear. We will do that by showing the two conditions of linearity:

1. $f(x_0) + f(x_1) = f(x_0 + x_1)$.

   ✗

2. $\lambda \in \mathbb{R}$, $\lambda f(x_0) = f(\lambda x_0)$.

Condition 1: By the definition of pointwise addition in F, we know that $(f_1) + (f_2) = (f_1 + f_2)$. Hence, we know $f(x_0) + f(x_1) = f(x_0 + x_1)$. ✗
Therefore Condition 1 is true.

Condition 2:

To show that Condition 2 is true, we must show
$$\lambda f(x_0) + \lambda f(x_1) = f(\lambda x_0) + f(\lambda x_1).$$
Using the definition of scalar multiplication in F, we know that
$$\lambda f(x_0) = f(\lambda x_0) \text{ and } \lambda f(x_1) = f(\lambda x_1).$$ ✗
Substituting these values into the lefthand side yields the righthand side. Therefore Condition 2 holds.

Hence, $f(x_0) + f(x_1)$ is true as conditions 1 and 2 hold.

d. Consider the problem statement. To be a subspace, one of the conditions is that $0 \in V$. This means that the subspace must consist of a "zero" element. Since the subset $f(0) = 1$ does not contain a "zero" element, it is not a subspace. ✓

e. Consider the problem statement. To be a subspace, three conditions must be met:

1. $0 \in V$
2. $v \in V, w \in V, v+w \in V$
3. $\lambda \in \mathbb{R}, v \in V, \lambda v \in V$.

Condition 1

Since $f(-1) \cdot f(1) = 0$ has a mapping to a "zero" element such that anything added to it results in itself, Condition 1 is true. ✓

Condition 2. Since the set contains only $f(-1) \cdot f(1)$, to test this we see that
$$f(-1) \cdot f(1) + f(-1) \cdot f(1) = 0 + 0 = 0 \in \text{ of the subset.}$$ Hence condition 2 holds. ✗

Condition 3: for some $\lambda \in \mathbb{R}$, we see that $\lambda(f(-1) \cdot f(1)) = \lambda(0)$. Since we know $0 \cdot$ any number is 0, we know $\lambda(0) = 0$, which is an element of the subset. ✓

Hence, since conditions 1, 2, 3 hold, $f(-1) \cdot f(1) = 0$ is a subspace. ✗

Problem 3. *5pts.*

Let

$$e_1 = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \quad e_4 = \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \quad e_5 = \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}, \quad e_6 = \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}.$$

Consider the 6×6 matrix

$$A = \left(2e_5 \middle| e_3 \middle| 3e_6 \middle| 5e_4 \middle| e_1 \middle| e_2 \right).$$

Calculate A^{30} and A^{31} efficiently.

I expect the numbers in your matrix to be written in exponent form;

don't calculate things like 2^{13}.

You can write your matrices in terms of their columns like I have done.

$2: \ (2e_1 \mid 3e_6 \mid 3e_2 \mid 25e_4 \mid 2e_5 \mid e_3) \to 3: \ (2^2 e_5 \mid 3^2 e_2 \mid 3e_3 \mid 5^3 e_4 \mid 2e_1 \mid 3e_6)$

$\to 4: \ (2^2 e_1 \mid 3e_3 \mid 3^2 e_6 \mid 5^4 e_4 \mid 2^2 e_5 \mid 3e_2) \to 5: \ (2^3 e_5 \mid 3^2 e_6$

$A^{30} = (2^{15} e_1 \mid 3^{10} e_2 \mid 3^{10} e_3 \mid 5^{30} e_4 \mid 2^{15} e_5 \mid 3^{10} e_6)$

$A^{31} = (2^{16} e_5 \mid 3^{10} e_3 \mid 3^{11} e_6 \mid 5^{31} e_4 \mid 2^{15} e_1 \mid 3^{10} e_2)$

Problem 4

2 (a) [3pts.] Prove that the following tuple spans $\mathbb{R}^3$.

$$\big((1,0,0),\ (1,1,0),\ (1,1,1)\big)$$

0 (b) [4pts.] Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For this particular example, describe $\ker T$, and prove your claim.

a. Consider the problem statement. We want to show that the tuple spans $\mathbb{R}^3$. We see that if we take $\lambda_1(1,0,0)$, we can reach any number in $\mathbb{R}^1$. We see that if we take $\lambda_1(1,0,0) + \lambda_2((1,1,0)-(1,0,0))$, we can reach any number in $\mathbb{R}^2$. We see that $\lambda_1(1,0,0) + \lambda_2((1,1,0)-(1,0,0)) + \lambda_3((1,1,1)-(1,1,0))$ allows us to reach any number in $\mathbb{R}^3$. Hence, a linear combination of $((1,0,0), (1,1,0), (1,1,1))$ spans all of $\mathbb{R}^3$, which means that the tuple spans $\mathbb{R}^3$.

not precise, need to pick λs

b. $\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$? X

The $\ker(T)$ is spanned by the line $\mu(0,1,0)$, where $\mu \in \mathbb{R}$. We will show this via the definition of $\ker(T) := \{v \in V : T(v) = 0\}$. If we view the transformation matrix, we see that any value in the middle column (x_2) is sent to 0 as it takes $0(x_2) + x_2 - x_2 = 0$. Furthermore, we see that the values are multiplied by a scalar in columns 1 and 3: $(x_1) + x_1 + 0 = 2x_1$, $-x_3 + 0 - x_3 = -2x_3$. Hence, the only $v \in V$ that maps $T(v) = 0$ is the line spanned by $\mu \in \mathbb{R}$, $\mu(0,1,0)$. This also contains the 0 vector, $(0,0,0)$, which also works $T(0) = 0$.

only showed $\subseteq$ (incorrectly)

Problem 5.

Suppose V and W are vector spaces over $\mathbb{R}$, that $S : V \to W$ is a linear transformation, and that $(v_1, v_2, \ldots, v_n)$ is a tuple of vectors in V.

Always true or sometimes false (i.e. depends on V, W, S, etc.)?

(a) [2pts.] If $(v_1, v_2, \ldots, v_n)$ spans V,
then $(S(v_1), S(v_2), \ldots, S(v_n))$ is linearly independent and $\ker S = \{0\}$.

F ✓

(b) [2pts.] If $\ker S = \{0\}$ and $(S(v_1), S(v_2), \ldots, S(v_n))$ is linearly independent,
then $(v_1, v_2, \ldots, v_n)$ is linearly independent.

T ✓

(c) [2pts.] If $\operatorname{im} S = W$, then $(S(v_1), S(v_2), \ldots, S(v_n))$ spans W.

T ✗

(d) [2pts.] If $(v_1, v_2, \ldots, v_n)$ is linearly independent,
then $(S(v_1), S(v_2), \ldots, S(v_n))$ is linearly independent.

F ✓

For each part, respond with "T" or "F" or leave it blank.

Correct $\to$ 2pts. Blank $\to$ 1pt. Incorrect $\to$ 0pts.

No justification is required, and anything else that is written will be ignored.