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Spring 2019
UCLA

Linear Algebra, Math 115A
Final Exam, June 12

Last Name:

First Name:

Student ID:

Herewith I confirm that the above information is correct and that I have read and understood the instructions below:

Signature:

Instructions:

Do not open the exam until instructed to. You will have 180 minutes to complete the exam. Please print your name and student ID number above. Please write your initials on every sheet as indicated. You may **not** unstaple the exam.

You may **not** use calculators, cell phones, books, notes, or any other material to help you. Cell phones need to be **silenced** and stored in your bag. Bags need to be placed near the black board or a wall.

You may use any available space on the exam for scratch work. In particular, you can continue an exercise on the back of the page. There is **scratch paper** at the end of the exam, if you need more, please ask one of the proctors. Please write your name and UID on these sheets. If you continue an exercise on a different page, please indicate.

You must **show your work** to receive full credit, unless stated explicitly otherwise.

Question	1	2	3	4	5	6	7	8	9	10	$\sum$
Points											
Total	6	5	6	7	5	8	8	7	6	9	67

Exercise 1 **(2+4 points)**

Consider the map $T\colon M_{2\times 2}(\mathbb{Q}) \to M_{2\times 2}(\mathbb{Q})$, $T(A) = AX - XA$ where

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

a) Please show that T is linear.

b) Please find the matrix representation $[T]_B^B$ of T with respect to the ordered basis

$$B = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$$

Exercise 2 (3+2 points)

a) Let V be a vector space over the field $\mathbb{F}$ and $U \subseteq V$ be a subspace. Please show that the set

$$\{T \in \mathcal{L}(V) \mid \operatorname{im}(T) \subseteq U\}$$

is a subspace of the vector space $\mathcal{L}(V)$ of $\mathbb{F}$-linear maps from V to V.

b) Let $n \in \mathbb{N}$ and suppose that $T_i \in \mathcal{L}(\mathbb{F}^n)$ are given by

$$T_i \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $i = 1, \ldots, n$. Please show that $T_1, \ldots, T_n$ are linearly independent.

Exercise 3 **(3+3 points)**

Let V be a finite dimensional vector space over the field $\mathbb{F}$ and $T\colon V \to V$ linear. Please show that the following statements are equivalent:

a) $\ker(T) = \operatorname{im}(T)$

b) $T^2 = 0$, $\dim V$ is even and $\operatorname{rank}(T) = \frac{\dim V}{2}$

Exercise 4 (2+5 points)
For $x \in \mathbb{R}$ let $U_x \subseteq \mathbb{R}^3$ be the subspace

$$U_x = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ x \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 1 \\ 1 \end{pmatrix} \right\}.$$

a) Please find all $x \in \mathbb{R}$ such that $\dim U_x < 3$.

b) Please find for every $x \in \mathbb{R}$ with $\dim U_x < 3$ a basis of U_x which is *orthonormal* with respect to the standard inner product on $\mathbb{R}^3$.

Exercise 5 (5 points)

Let $\mathbb{R}[X]_{\leq 2}$ denote the vector space of polynomials of degree at most 2. Please show that

$$\langle f, g \rangle = \sum_{i=0}^{2} f(i)g(i)$$

defines an inner product on $\mathbb{R}[X]_{\leq 2}$.

Exercise 6 (3+2+3 points)

Let V be a finite dimensional vector space over the field $\mathbb{F}$ with $\dim V = n$, and let $T\colon V \to V$ be linear. Recall that $T^m = \underbrace{T \circ \ldots \circ T}_{m \text{ times}}$ for $m \geq 1$.

a) Let $m \in \mathbb{N}$. Suppose that $v \in V$ is an eigenvector of T with respect to the eigenvalue λ. Please show by induction that v is also an eigenvector of T^m and compute the corresponding eigenvalue.

b) Suppose that $T^k = 0$ for some $k \in \mathbb{N}$ and suppose that $\lambda \in \mathbb{F}$ is an eigenvalue of T. Please show that $\lambda = 0$.

c) Suppose that $T^k = 0$ for some $k \in \mathbb{N}$ and $T(v) \neq 0$ for some $v \in V$. Is T diagonalizable? Please explain.

Exercise 7 **(3+3+2 points)**

Let $\mathbb{R}[X]_{\leq 2}$ denote the vector space of polynomials of degree at most 2 and suppose that

$$A = \begin{pmatrix} 1 & -1 & -5 \\ 2 & -1 & -8 \\ -1 & 2 & 7 \end{pmatrix}$$

is the matrix representation of the linear map

$$T\colon \mathbb{R}[X]_{\leq 2} \to \mathbb{R}[X]_{\leq 2}$$

with respect to the basis

$$B = (1 + X, 2X, 2 - X^2),$$

i.e. $A = [T]_B^B$.

a) Please find a basis for the image of T.

b) Please find a basis for the kernel of T.

c) Please compute $T(1)$.

Exercise 8 **(3+4 points)**

Let V be a vector space over the field $\mathbb{F}$ of dimension $\dim V = 3$ and let $T\colon V \to \mathbb{F}$ be linear.

a) Please show that T is not injective.

b) Suppose that T is surjective. Please show that there is a basis B_V of V and $\lambda \in \mathbb{F}$ such that

$$\lambda \neq 0 \quad \text{and} \quad [T]_{B_V}^{B_{\mathbb{F}}} = \begin{pmatrix} 0 & 0 & \lambda \end{pmatrix}$$

where $B_{\mathbb{F}}$ is the standard basis of $\mathbb{F}$ consisting of $e_1 = 1$.

Exercise 9 **(5+1 points)**

Let V be a finite dimensional vector space over the field $\mathbb{F}$. Let $U_1 \subseteq U_2$ be subspaces of V.

Please show that there is a subspace $W \subseteq V$ with $U_2 \cap W = U_1$ and $U_2 + W = V$. Please show that your choice of W indeed has the desired properties.

Please give an example for W in the case $V = \mathbb{R}^3$, $U_1 = \operatorname{span}(e_1)$, $U_2 = \operatorname{span}(e_1, e_2)$. No further justification required.

Hint. Start with the example. Construct a basis B of V which contains a basis of U_1 and a basis of U_2 as subsets. Define W to be the span of a suitable subset of B.

Exercise 10 **(3+4+2 points)**

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & -1 \\ -1 & 0 & 0 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$$

and let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map $T(x) = Ax$.

a) Please compute the eigenvalues of T and determine their algebraic multiplicities.

b) Show that T is diagonalizable and find an eigenbasis B.

c) Find a matrix Q such that $D = Q^{-1}AQ$ is diagonal and compute D.
[Notice that you do not need to compute Q^{-1}.]